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Quantum and braided group Riemannian geometry

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Abstract

We formulate quantum group Riemannian geometry as a gauge theory of quantum differential forms. We first develop (and slightly generalise) classical Riemannian geometry in a self-dual manner as a principal bundle frame resolution and a dual pair of canonical forms. The role of Levi-Civita connection is naturally generalised to connections with vanishing torsion and cotorsion, which we introduce. We then provide the corresponding quantum group and braided group formulations with the universal quantum differential calculus. We also give general constructions, for example, including quantum spheres and quantum planes.

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1. Introduction

One of the long-term motivations for noncommutative geometry is a theory of Riemannian geometry and gravity powerful enough to not break down in the quantum domain. The quantum groups approach to such 'Planck scale geometry' is initiated in [1] in the context of quantum phase spaces which are quantum groups. The axioms of a quantum group provide a way to unify the noncommutativity due to quantisation and the noncocommutativity or non-Abelianness of the coproduct due to curvature. To extend this programme to more realistic models, however, one needs a more general framework not limited to analogues of group manifolds. One may expect that quantum groups will still play a role as symmetries

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or quantum gauge groups and that they will ensure a plentiful supply of examples such as q-deformed homogeneous spaces. A further and more general motivation is the q-deformation of geometric structures to provide a new regularisation parameter 'q' whether or not it is related to Planck's constant. In some settings, q, in fact, generalises the role of -1 in bose-fermi statistics, so that this quantum geometry effectively generalises supergeometry as well [2]. We refer to the text [3] for much of this background.

Motivated in part by such considerations, quantum group gauge theory has already been introduced in [4] and several follow-up papers [5–8]. It has also been extended to more general settings [9,10]. On the other hand, Riemannian geometry has not yet been included in this set-up. In the present paper we provide a start to this programme by giving a fresh formulation of classical Riemannian geometry as a gauge theory (in Section 2) and proceeding (in Section 3) to the parallel definitions in the noncommutative or 'quantum' case. We limit our results to the 'free' or universal differential calculus $\Omega^1 M$ associated to any (not necessarily commutative algebra M), but the case of nonuniversal differential calculus can be treated similarly and will be developed in detail elsewhere.

Riemannian geometry as a gauge theory is obviously very well known, in the physics literature under the heading of 'vierbeins' and in the mathematics literature as the theory of so-called 'linear' or GL_n frame connections [11]. The key to our proposal for a quantum group version is a 'clean' reformulation in Section 2 of the classical theory based on general Lie groups and differential forms, and of the Levi-Civita connection in a slightly generalised but 'self-dual' manner as one with vanishing torsion and 'cotorsion'. We recall that usually one views the tangent bundle of a manifold as associated to the bundle of linear frames (a certain GL_n principal bundle), and the Levi-Civita connection as induced from a gauge connection on it. Similarly one has O_n and affine frame bundles, usually treated separately [11]. Key to the theory is the canonical one-form θ on the frame bundle, which is known (see the appendix of [5]) to fully characterise it. We first extend this point of view by providing derivations of the torsion and curvature tensors (the main formulae of Riemannian geometry) associated to a *frame resolution* (P, G, V, θ) (where P is a principal G-bundle and V a representation of G), independent of its detailed form. We have not found such a nonspecific (and form based) treatment elsewhere, although [12] comes close in some points (notably Proposition 2.3). In particular, Proposition 2.5 interprets the torsion tensor as the difference $T = \mathrm{d} - \nabla$ between the covariantised exterior derivative on 1-forms and the usual exterior derivative. We then show that a dual frame resolution (P, G, V^*, γ) is equivalent to a (not-necessarily symmetric) metric tensor, which is our new 'self-dual' way of thinking about Riemannian geometry. The dual theory has the roles of V, θ and V^*, γ interchanged, which also transposes the metric (so the usual symmetric metric is the self-dual case). We show that the vanishing of the *cotorsion* (the torsion in the dual frame resolution) is a skew version of metric compatibility. This gives a natural generalisation of Levi-Civita connections as ones with zero torsion and zero cotorsion.

Proceeding to the quantum group case, the main problem of quantum group Riemannian geometry is what quantum group? Surely not $SO_q(n)$ or $GL_q(n)$ in general as 'quantum spaces' are not in general 'manifolds' based on patching $\mathbb{R}^n_q$. In general they are algebras without necessarily even a dimension n. Based on our formulation of the classical theory in Section 2, we propose in Section 3 to consider as 'manifold' a possibly noncommutative algebra M equipped with a differential calculus $\Omega^1(M)$ (or higher forms) as in quantum group gauge theory, and to consider any 'quantum frame resolution' of $(M, \Omega^1(M))$. This is any quantum principal bundle over M equipped with certain further structure. In this way, we are not forced to choose which quantum group will play the role of structure group of the frame bundle. Any choice of resolution will serve the purpose of expressing the quantum (co)tangent bundle as an associated vector bundle and allow the first steps of quantum group Riemannian geometry as an application of the gauge theory of associated bundles already in [4]. The second and related problem is that one cannot expect the quantum metric to be symmetric (it typically has some form of q-symmetry). Again motivated by our formulation of the classical theory, we drop such a consideration and formulate the metric as a dual frame resolution.

As a modest first application of the quantum group version, we show in Section 4 that the well-known 'parallelisation isomorphism' $H \otimes \ker \epsilon \cong \Omega^1 H$ used in the theory of bicovariant differential calculi on a quantum group H can be understood 'geometrically' as induced by a frame bundle resolution. We also show (Proposition 4.2) how Fourier transformation on a finite-dimensional Hopf algebra can be interpreted as inducing a quantum metric on it. We then turn to more general classes of examples, including a frame resolution for any quantum homogeneous space (such as the q-sphere) obtained as a homogeneous bundle $P \to H$, and a frame bundle resolution for any braided group B (such as the quantum plane $M = \mathbb{R}^n_q$) obtained as its bosonisation. The structure group in the quantum plane example is $GL_q(n)$, i.e. this is the natural 'flat space' example.

The paper concludes with an appendix detailing the corresponding theory with braid statistics, proven diagrammatically. This is for completeness as a supplement to the diagrammatic or braided group gauge theory in [9,10]. The still more general coalgebra bundle case [9] is a direction for further work.

Finally, although we will indicate briefly how the results generalise to nonuniversal differential calculi, the detailed theory in this case must await the theory of associated bundles with nonuniversal calculus. The first step, namely the construction of natural nonuniversal calculi $\Omega^1(P)$ from $\Omega^1(M)$ and nonuniversal calculi on the fibre, has recently been obtained in [8]. However, the extension of this to associated bundles is a separate and quite involved project, to be considered in a sequel.

Our quantum groups approach should ultimately link up with other approaches to noncommutative geometry, notably with that of Connes [13]. There one considers only vector bundles (as projective modules) and abstractly defined covariant derivations, and not principal bundles and connection forms (which really need the quantum groups approach [4]). Some initial results relating at least gauge theory in the two approaches are in [14]. Among other papers, Heckenberger and Schmüdgen [15] have recently given examples in the vector bundle and 'covariant derivation' approach with the base M a standard quantum group such as $SO_q(n)$ (with nonuniversal differential calculus); it should ultimately be possible to understand this as part of a general theory of the type presented here. Finally, another attempt at frame bundles can be found in [16]. By contrast, we have been motivated by the appendix of the earlier paper [5] and our approach includes basic examples such as q-spheres and q-planes.

2. Riemannian geometry with respect to a frame resolution

Our reformulation of Riemannian geometry is based on the following lemma about associated bundles. It can be considered as implicit in [11], though perhaps not stated explicitly. Let M be a manifold and $C(M)$ functions on M. We work in a smooth setting throughout the section. Let $\pi : P \to M$ be a principal bundle over M with structure group G, and let V be a left G-module. We recall that there is an associated vector bundle $\mathcal{E} = P \times_G V$ consisting of equivalence classes in $P \times V$ under the relation $(p \cdot g, v) \sim (p, g \cdot v)$ for all $p \in P, v \in V$ and $g \in G$. The projection to M is $\pi(p, v) = \pi(p)$ and the fibre over $x \in M$ is isomorphic to V. We recall that sections of $\mathcal{E}$ are maps $M \to \mathcal{E}$ such that following with the projection to M gives the identity, and that sections may be identified with pseudotensorial (i.e. G-equivariant) functions on P:

$$\Gamma(\mathcal{E}) \cong C_G(P, V).$$

We extend this now to vectors and forms. We denote by $\Omega^{-1}(M)$ the space of vector fields on M and, later on, by $\Omega^1(M)$ the space of 1-forms on M. These are sections of the tangent bundle TM and the cotangent bundle T^*M, respectively. A differential from on P is called *tensorial* if it is equivariant and 'horizontal' in the sense that it vanishes on the vertical vector fields corresponding to the action of G on P. Finally, we recall that a bundle map between bundles over M means a map between the total spaces forming a commutative triangle with the projections to M.

Lemma 2.1. *The space $\Omega^1_{\mathrm{tensorial}}(P, V)$ of V-valued tensorial 1-forms is in correspondence with the space of bundle maps $TM \to \mathcal{E}$. These in turn are in correspondence with $C(M)$-module maps*

$$\Omega^{-1}(M) \to C_G(P, V).$$

Proof. The fibre-wise formulae are as in [11]: given θ which is G-equivariant and horizontal, we define a map $\tilde{\theta} : T_x M \to \mathcal{E}_x$ by $\tilde{\theta}(X_x) = [(p, \theta(\tilde{X}))]$ where $p \in P$ is such that $\pi(p) = x$ and $\tilde{X}$ is any vector field on P with horizontal projection $\pi_*(\tilde{X}) = X$ at x (i.e. any local lift). One may then check that this construction extends smoothly: in each open set U choose a local section $\sigma : U \to P$ to specify the lifts. □

An example is provided by the frame bundle. We recall that a frame at $x \in M$ is a linear isomorphism $\mathbb{R}^n \to T_x M$ (i.e. a choice of basis of $T_x M$). The frame bundle $P = FM$ is a principal bundle over M with structure group $G = GL_n$ and fibre over x given by the set of all frames at x. In a local patch where the bundle is trivial, we denote by

$\pi_2(p) : \mathbb{R}^n \to T_{\pi(p)}M$ the corresponding frame. The canonical 1-form $\theta \in \Omega^1_{\rm tensorial}(P, \mathbb{R}^n)$ is locally defined by

$$\theta_p(X) = \pi_2(p)^{-1}\pi_*(X)(\pi(p))$$

for all $X \in TP$. One may check that it is tensorial and globally defined. Here $V = \mathbb{R}^n$ as a GL_n-module and the map in Lemma 2.1 in this case is an isomorphism $TM \cong \mathcal{E} = FM \times_{GL_n} \mathbb{R}^n$. Thus Lemma 2.1 tells us that the tangent bundle is an associated vector bundle to the frame bundle.

Moreover, the fact that θ characterises the frame bundle as observed in [5] is now recovered as an immediate corollary of Lemma 2.1:

Corollary 2.2. *If (P, θ), (P', θ') are two GL_n-bundles over M equipped with equivariant tensorial 1-forms with values in $\mathbb{R}^n$ and such that their induced maps in Lemma 2.1 are isomorphisms, then $P \cong P'$ and θ' may be identified with θ.*

Proof. Clearly $P \times_{GL_n} \mathbb{R}^n \cong TM \cong P' \times_{GL_n} \mathbb{R}^n$ as associated GL_n bundles. Since the actions on $V = \mathbb{R}^n$ in both cases are the same fundamental representation of GL_n, which is faithful, the structure functions of P and P' are equivalent, i.e. they are isomorphic principal bundles. □

Taking $P = FM$ and θ the canonical form, any other P', θ' with the same structure group GL_n is therefore isomorphic. But one *could* have different examples with different structure groups. For example, one may take the bundle of affine frames with structure group $\mathbb{R}^n \rtimes GL_n$ or, for a manifold admitting a metric, the bundle of orthogonal frames with structure group O_n. We call any (P, G, V, θ) inducing an isomorphism via Lemma 2.1 a *frame resolution* of the tangent bundle. We now extend these ideas further, to include differential forms. As a dualisation of Lemma 2.1, we have (cf. [12, Section 11.14]):

Proposition 2.3. *Let V in the setting of Lemma 2.1 be finite-dimensional and V^* its dual as a G-module. $\theta \in \Omega^1_{\rm tensorial}(P, V)$ are in correspondence with bundle maps $\mathcal{E}^* \to T^*M$ and, at the level of sections, with $C(M)$-module maps*

$$C_G(P, V^*) \to \Omega^1(M).$$

Proof. We dualise Lemma 2.1 in a straightforward manner (fibrewise). Here $\mathcal{E}^* = P \times_G V^*$ is the fibre-wise dual. Explicitly, given θ, a V^*-valued equivariant function ϕ maps to the one form which at x has values $(\pi^*)^{-1}\phi_p \cdot \theta_p$ where we chose any $p \in \pi^{-1}(x)$ and identify θ_p as in the image of $\pi^* : T_xM \to T_pP$. Here $\cdot$ denotes the evaluation of V^* with V. A similar correspondence has been pointed out to us in [12, Section 11.14]; for our purposes we need the correspondence quite explicitly in the form just given. □

In the case of the frame bundle, this is an isomorphism and expresses the cotangent bundle as an associated vector bundle $T^*M \cong FM \times_{GL_n} \mathbb{R}^{n*}$. Similarly for any frame resolution

(P, G, V, θ) we see that both 1-forms and vector fields on M may then be expressed as V or V^*-valued equivariant functions on P. In a similar manner, one has in this case.

$$\Omega^1(M) \underset{C(M)}{\otimes} \Omega^1(M) \cong \Omega^1_{\text{tensorial}}(P, V^*), \tag{1}$$

$$\Omega^1(M) \underset{C(M)}{\otimes} \Omega^{-1}(M) \cong \Omega^1_{\text{tensorial}}(P, V), \tag{2}$$

$$\Omega^2(M) \underset{C(M)}{\otimes} \Omega^1(M) \cong \Omega^2_{\text{tensorial}}(P, V^*), \tag{3}$$

$$\Omega^2(M) \underset{C(M)}{\otimes} \Omega^{-1}(M) \cong \Omega^2_{\text{tensorial}}(P, V), \tag{4}$$

etc. The canonical form $\theta \in \Omega^1_{\text{tensorial}}(P, V)$ corresponds to the constant section of $\Omega^{-1}(M) \otimes_{C(M)} \Omega^1(M)$ given over each point x by the canonical element of $T_x M \otimes T^*_x M$.

Now, when P is equipped with a connection, it induces a connection on $\mathcal{E}$ and hence a corresponding covariant derivative D on sections of $\mathcal{E}$. In the present case it means a covariant derivative on vector fields, 1-forms, etc. when these are viewed via the above isomorphisms as sections of suitable associated vector bundles. In this way, one obtains the usual formulae of Riemannian geometry but now as a gauge theory on *any* frame resolution. As such, many of the formulae are, in fact, more natural.

Lemma 2.4. *Let (P, G, V, θ) be a frame resolution of a manifold M, and ω a connection on P. Define the induced $\nabla : \Omega^1(M) \to \Omega^1(M) \otimes_{C(M)} \Omega^1(M)$ as the covariant derivative $D : C_G(P, V^*) \to \Omega^1_{\text{tensorial}}(P, V^*)$ viewed under the above isomorphisms. Then ∇ is a derivation with respect to multiplication by functions on M. Moreover,*

$$\pi^* \nabla_X f = \pi^* L_X f - \phi \cdot \mathcal{L}_{\tilde{X}} \theta, \quad \mathcal{L} = L + \omega$$

for $\phi \in C_G(P, V^*)$ corresponding to $f \in \Omega^1(M)$ and $\tilde{X}$ any lift of a vector field X on M. Here L denotes the Lie derivative and $\cdot$ the evaluation of V^* with V.

Proof. Here $\pi^* f = \phi \cdot \theta$ as in Proposition 2.3, and similarly $\pi^* \nabla_X f = (D_{\tilde{X}} \phi) \cdot \theta = (\tilde{X}(\phi) + \omega_{\tilde{X}} \phi) \cdot \theta$ by the similar isomorphism (1). Note that the covariant derivative on $C_G(P, V^*)$ is defined by

$$D\phi = (\mathrm{id} - \Pi_\omega)\, \mathrm{d}\phi = D\phi - \tilde{\omega}(\phi) = \mathrm{d}\phi + \omega\phi,$$

since ϕ is equivariant in the sense $\tilde{\xi}(\phi) = -\xi\phi$ for all $\xi \in \mathfrak{g}$, the Lie algebra of G. Here $\tilde{\xi}$ is the vector field on P induced by the action of ξ and Π_ω is the projection on $\Omega^1(P)$ corresponding to the connection form ω. These steps are the standard definition of $D\phi$, as in [11] (we recall them explicitly since we use the quantum group version in Section 3). We then evaluate against any lift $\tilde{X}$ of X to a vector field on P. Here $D_{\tilde{X}}\phi$ is manifestly independent of the choice of lift since $\omega_{\tilde{\xi}} = \xi$ for any connection form, and ϕ is equivariant. Also, if $g \in C(M)$, then gf corresponds to $(\pi^* g)\phi$. Hence $\pi^* \nabla g f = (D((\pi^* g)\phi)) \cdot \theta = \pi^*(\mathrm{d}g)\phi \cdot \theta + (\pi^* g)(D\phi) \cdot \theta$, i.e. ∇ is a derivation. Finally, writing $\tilde{X}(\phi) = L_{\tilde{X}}\phi$ and the Leibniz property of the Lie derivative (and moving ω to act on V rather than V^* by $(\xi\phi) \cdot \theta = -\phi \cdot \xi\theta$) gives the alternative expression for $\nabla_X f$ as stated. □

We call the operation $\mathcal{L} = L + \omega$ on form-sections the *covariant Lie derivative*. We see that $\mathcal{L}\theta$ measures the deviation of the covariant derivative from the Lie derivative on forms. Finally, we define the covariant derivative on vector fields by

$$\langle \nabla_X Y, f\rangle = X(\langle Y, f\rangle) - \langle Y, \nabla_X f\rangle \tag{5}$$

for all vector fields X, Y on M and all $f \in \Omega^1(M)$.

Proposition 2.5. *Define the torsion tensor T as the $(1,2)$-tensor on M corresponding to $D\theta \in \Omega^2_{\mathrm{tensorial}}(P, V)$ under the above isomorphisms. Then*

$$\nabla \wedge f = \mathrm{d}f - \langle T, f\rangle$$

for all $f \in \Omega^1(M)$, which is equivalent to the usual definition of the torsion tensor.

Proof. We first show that the equation shown for T is equivalent to the usual definition

$$\nabla_X Y - \nabla_Y X = [X, Y] + T(X, Y)$$

for all vector fields X, Y on M. Using (5), this is equivalent to

$$X(\langle Y, f\rangle) - \langle Y, \nabla_X f\rangle - Y(\langle X, f\rangle) + \langle X, \nabla_Y f\rangle - \langle [X, Y], f\rangle = \langle T(X, Y), f\rangle.$$

On the other hand, using the notation i_X for interior product with forms, L_X for Lie derivative, and the identities $L_X = i_X\,\mathrm{d} + \mathrm{d}i_X$, $[L_X, i_Y] = i_{[X,Y]}$, we have

$$i_Y i_X\,\mathrm{d}f = -i_Y\,\mathrm{d}i_X f + i_Y L_X f = X(\langle Y, f\rangle) - Y(\langle X, f\rangle) - \langle [X, Y], f\rangle \tag{6}$$

as required. Here $i_Y i_X(\nabla \wedge f) = (\nabla \wedge f)(X, Y) = \langle Y, \nabla_X f\rangle - \langle X, \nabla_Y f\rangle$.

We now let T correspond to $D\theta$ by (4). Equivalently, $T(X, Y)$ is a vector field corresponding under the induced isomorphism in Lemma 2.1 to $(D\theta)_{\tilde{X},\tilde{Y}} \in C_G(P, V)$. Hence $\pi^*\langle T(X, Y), f\rangle = \phi \cdot (D\theta)_{\tilde{X},\tilde{Y}} = i_Y i_X \phi \cdot D\theta$.

Once these isomorphisms are understood, the computation in the tensorial form language is immediate: $\pi^*\nabla \wedge f = (D\phi)\wedge\theta = (\mathrm{d}\phi)\wedge\theta + \omega\phi\wedge\theta = \mathrm{d}(\phi\cdot\theta) - \phi\cdot\mathrm{d}\theta - \phi\cdot\omega\wedge\theta = \pi^*\,\mathrm{d}f - \phi\cdot D\theta$ as required. □

The operation $\nabla\wedge$ is the *covariant exterior derivative* and we see that the torsion T measures its difference from the usual exterior derivative on 1-forms. We can similarly treat the Riemannian curvature in terms of differential forms and sections.

Proposition 2.6. *Define the curvature tensor of ∇ by Rf the 1-form corresponding to $\Omega\phi$ where $\Omega \in \Omega^2_{\mathrm{tensorial}}(P, \mathfrak{g})$ is the curvature of the connection form ω and $f \in \Omega^1(M)$ corresponds to ϕ. Then*

$$R(X, Y)f = [\nabla_X, \nabla_Y]f - \nabla_{[X,Y]}f$$

for all vector fields X, Y on M.

Proof. It is easy to check that the stated formula for $R(X, Y)$ acting on forms is equivalent to its usual definition as an operator on vector fields, when the two are related by $\langle Z, R(X, Y)f\rangle = -\langle R(X, Y)Z, f\rangle$ for all X, Y, Z and f. We use (5) repeatedly to establish this. Next, from the definition of ∇ above, $\pi^*(\nabla_X\nabla_Y f - \nabla_Y\nabla_X f - \nabla_{[\tilde{X},\tilde{Y}]}f) = (([D_{\tilde{X}}, D_{\tilde{Y}}] - D_{[\tilde{X},\tilde{Y}]})\phi)\cdot\theta$. To compute this, note that

$$\begin{aligned}\tilde{X}(\omega_{\tilde{Y}}\phi) &= L_{\tilde{X}} i_{\tilde{Y}}(\omega\phi) = i_{\tilde{Y}} L_{\tilde{X}}(\omega\phi) + i_{[\tilde{X},\tilde{Y}]}\omega\phi\\ &= \omega_{\tilde{Y}}\tilde{X}(\phi) + i_{\tilde{Y}}(L_{\tilde{X}}\omega)\phi + \omega_{[\tilde{X},\tilde{Y}]}\phi\\ &= \omega_{\tilde{Y}}\tilde{X}(\phi) + i_{\tilde{Y}} i_{\tilde{X}}(\mathrm{d}\omega)\phi + i_{\tilde{Y}}(\mathrm{d}i_{\tilde{X}}\omega)\phi + \omega_{[\tilde{X},\tilde{Y}]}\phi\\ &= \omega_{\tilde{Y}}\tilde{X}(\phi) + i_{\tilde{Y}} i_{\tilde{X}}(\mathrm{d}\omega)\phi + \tilde{Y}(\omega_{\tilde{X}}\phi) - \omega_{\tilde{X}}\tilde{Y}(\phi) + \omega_{[\tilde{X},\tilde{Y}]}\phi.\end{aligned}$$

We used the usual formulae for $[L, i]$ and $L = i\,\mathrm{d} + \mathrm{d}i$, this time in P. Hence

$$\begin{aligned}[D_{\tilde{X}}, D_{\tilde{Y}}]\phi &= D_{\tilde{X}}(\tilde{Y}(\phi) + \omega_{\tilde{Y}}\phi) - (X\leftrightarrow Y)\\ &= \tilde{X}(\tilde{Y}(\phi)) + \omega_{\tilde{X}}\tilde{Y}(\phi) + \tilde{X}(\omega_{\tilde{Y}}\phi) + \omega_{\tilde{X}}\omega_{\tilde{Y}}\phi - (X\leftrightarrow Y)\\ &= [\tilde{X}, \tilde{Y}]\phi + \omega_{[\tilde{X},\tilde{Y}]}\phi + i_{\tilde{Y}} i_{\tilde{X}}(\mathrm{d}\omega + \omega\wedge\omega)\phi\\ &= \Omega_{\tilde{X},\tilde{Y}}\phi + D_{[\tilde{X},\tilde{Y}]}\phi.\end{aligned}$$

as required. □

One may also define the exterior covariant derivative on 2-forms as corresponding to D on $\Omega^1(P, V^*)$, and then $\nabla\wedge\nabla\wedge f = R\wedge f$ holds, cf. [12]. Moreover, to complete the picture, we can also consider the metric under

$$\Omega^1(M) \underset{C(M)}{\otimes} \Omega^1(M) \cong C_G(P, V^*) \underset{C(M)}{\otimes} C_G(P, V^*) \cong C_G(P, V^*\otimes V^*) \tag{7}$$

as an equivariant function on P with values in $V^*\otimes V^*$. The first isomorphism here is that in Proposition 2.3 applied to each tensor factor over $C(M)$. The second is given by pointwise product in P and is clearly an isomorphism for trivial bundles (where $C_G(P, V^*) = C(M, V^*)$, etc.), and hence holds generally by local triviality of a general bundle.

Proposition 2.7. *Let ∇_X on any $g\in\Omega^1(M)\otimes_{C(M)}\Omega^1(M)$ be defined as corresponding to $D: C_G(P, V^*\otimes V^*)\to\Omega^2_{\text{tensorial}}(P, V^*\otimes V^*)$ under the above isomorphisms. Then ∇_X is the extension of ∇_X on $\Omega^1(M)$ to the tensor product as a derivation.*

Proof. We make a similar computation to that in Lemma 2.4. Thus, if $g\in\Omega^1(M)\otimes_{C(M)}\Omega^1(M)$ is viewed as $\eta\in C_G(P, V^*\otimes V^*)$, then

$$\begin{aligned}\pi^*\nabla_X g &= (D_{\tilde{X}}\eta)\cdot(\theta\otimes\theta) = \pi^* L_X g - \eta\cdot\mathcal{L}_{\tilde{X}}(\theta\otimes\theta)\\ &= \pi^*((L_X g_\alpha)g^\alpha + g_\alpha L_X g^\alpha) - \eta\cdot(\mathcal{L}_{\tilde{X}}\theta\otimes\theta + \theta\otimes\mathcal{L}_{\tilde{X}}\theta),\end{aligned}$$

where $g = g_\alpha\otimes_{C(M)} g^\alpha$, say (summation understood). We used the derivation property of the Lie derivative and the fact that $\mathfrak{g}$ (where lies the output of ω) acts like a derivation on the tensor product representation $V^* \otimes V^*$. Finally, by the alternative expression for ∇ on $\Omega^1(M)$ in Lemma 2.4, we obtain

$$\nabla_X g = (\nabla_X g_\alpha) g^\alpha + g_\alpha \nabla_X g^\alpha \tag{8}$$

as stated. Note that evaluation against Y, Z and (5) shows that (8) is equivalent to the more conventional definition of $\nabla_X g$ as [11]:

$$(\nabla_X g)(Y, Z) = X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z). \qquad \square$$

Finally, in the case where the frame resolution is a trivial bundle, the form θ itself corresponds to the 'abstract n-bein' or V*-bein* $e \in \Omega^1(M, V)$ via $\theta(p) = \pi_2(p)^{-1}\pi^* e(p)$. The space $C_G(P, V^*) \cong C(M) \otimes V^*$ by $\phi(p) = \pi_2(p)\pi^*\psi$ where $\psi \in C(M, V^*)$ is a 'matter field' with values in V^*. These particular 'matter fields' correspond to 1-forms $f \in \Omega^1(M)$ by $f = \psi \cdot e$. Similarly, a metric corresponds to 'matter field' $\eta \in C(M, V^* \otimes V^*)$ by $g = \eta \cdot (e \otimes_{C(M)} e)$. Moreover, as usual, a connection ω corresponds to gauge field $A \in \Omega^1(M, \mathfrak{g})$ by

$$\omega(p) = \pi_2(p)^{-1}\pi^* A(p)\pi_2(p) + \pi_2(p)^{-1}\, \mathrm{d}\pi_2(p),$$

with the usual abuses of notation for the second term. This is how the quantities above look in terms of the usual 'matter fields' and 'gauge potentials'. Most manifolds do not admit trivial frame resolutions, but these are also the local formulae for each patch of a nontrivial frame resolution. One has similar formulae to the general case, for example,

$$\nabla_X f = L_X f - \psi \cdot \mathcal{L}_X e, \quad \mathcal{L} = L + A \tag{9}$$

if 1-form f corresponds to matter field ψ. Here $\mathcal{L}$ can be called the *covariant Lie derivative* on M. In particular, the Levi-Civita connection corresponds to (an example of) a gauge field leaving covariantly constant a matter field η with values in $V^* \otimes V^*$ and a 1-form e with values in V.

This completes our formulation of the main properties of the covariant derivative on forms in terms of connections on an arbitrary frame resolution. In the general case we could call a connection 'Levi-Civita' *with respect to a framing and a metric* if $D\theta = 0$ (torsion free) and $D\eta = 0$ (metric compatible). We should not expect existence and uniqueness, however, as this depends strongly on a particular choice of resolution. Instead, one should reverse the logic and think of the frame resolution as part of the input data and already playing in part the role of choice of a metric. The choice of this and a connection on it specifies a covariant derivative in lieu of a metric. In the standard O_n frame bundle case we know that connections correspond to metrics in the usual way, while a subgroup would only allow a subset of connections or a subset of corresponding metrics. Under this kind of correspondence, more general G are also possible, corresponding to locally preserving different kinds of 'generalised metrics'. For example, we may take G symplectic and symplectic forms in the role of metric. We could also consider manifolds equipped with, say, E_6 frame resolution or with resolution by infinite-dimensional groups, even without consideration of any metric.

Moreover, for any fixed G we can consider frame resolutions of $\Omega^1(M)$ via different representations V. For example, the spinor representation in the O_4 case leads ultimately to the spin bundle and Dirac operator. As a more novel application of this idea we consider now the representation V^* conjugate to any V. We show that this leads to a natural 'self-dual' formulation and slight generalisation of Riemannian geometry. We reformulate a (not necessarily symmetric) metric g as corresponding under (1) to $\gamma \in \Omega^1_{\rm tensorial}(P, V^*)$ and develop the theory symmetrically between γ and θ as defining conjugate frame resolutions, namely associated to V, V^*, respectively.

Corollary 2.8. *Given a frame resolution (P, G, V, θ), a 2-cotensor g is nondegenerate as a map $g : \Omega^{-1}(M) \cong \Omega^1(M)$ iff the corresponding γ makes (P, G, V^*, γ) into another frame resolution, called the dual frame resolution.*

Proof. Under the isomorphism s_θ of Corollary 2.2, an isomorphism $g : \Omega^{-1}(M) \to \Omega^1(M)$ (by $X \mapsto g(X, \ ))$ is equivalent to an isomorphism $\Omega^{-1}(M) \to C_G(P, V^*)$. It sends X to $\gamma_{\tilde X}$ (for any lift $\tilde X$) since $\gamma_{\tilde X} \cdot \theta = \pi^* g(X, \ )$. By Lemma 2.1, this is equivalent to a frame resolution (P, G, V^*, θ') where θ' is a tensorial V^*-valued 1-form on P such that $\Omega^{-1}(M) \to C_G(P, V^*)$ is given by mapping X to $\langle \theta', \tilde X \rangle$, i.e. $\theta' = \gamma$.

Note that, as a corollary, we have an explicit correspondence

$$f \in \Omega^1(M) \quad \leftrightarrow \quad \phi = \gamma_{\widetilde{g^{-1}(f)}} \tag{10}$$

in Proposition 2.4, etc., giving explicit formulae for the expressions there. □

We can also reverse the roles of θ and γ (and V^* and V) in all the above, regarding γ as the frame resolution and θ as corresponding to a generalised metric. From this point of view it is natural to replace $D\theta = 0$ and $\nabla g = 0$ by more symmetric 'self-dual' conditions $D\theta = 0$, $D\gamma = 0$.

Proposition 2.9. *Let (P, G, V, θ) be a frame resolution and g a nondegenerate 2-cotensor as 'generalised metric', viewed as a corresponding dual resolution (P, G, V^*, γ). We define the cotorsion form $\Gamma \in \Omega^2(M) \otimes_{C(M)} \Omega^1(M)$ by*

$$\Gamma(X, Y) = (\nabla_X g)(Y, \ ) - (\nabla_Y g)(X, \ ) + g(T(X, Y))$$

for vector fields X, Y on M. Then (i) *$\Gamma = g(T_\gamma)$ where T_γ is the torsion of γ in the dual frame resolution, and* (ii) *Γ corresponds under* (3) *to $D\gamma$.*

Proof. It is easy to see from the preceding corollary that $D\gamma$ corresponds under (3) to $g(T_\gamma)$. We prove part (ii). Using the identity (6) applied on P (and applied to lifts $\tilde X, \tilde Y$), we have

$$\begin{aligned}\langle Z, \Gamma(X, Y)\rangle &= (i_{\tilde Y} i_{\tilde X} D\gamma) \cdot \theta_{\tilde Z} \\ &= \tilde X(\gamma_{\tilde Y}) \cdot \theta_{\tilde Z} - \tilde Y(\gamma_{\tilde X}) \cdot \theta_{\tilde Z} - \gamma_{[\tilde X, \tilde Y]} \cdot \theta_{\tilde Z} \\ &\quad + (\omega_{\tilde X} \gamma_{\tilde Y}) \cdot \theta_{\tilde Z} - (\omega_{\tilde Y} \gamma_{\tilde X}) \cdot \theta_{\tilde Z}\end{aligned}$$

$$= \langle \nabla_X g_Y, Z\rangle - \langle \nabla_Y g_X, Z\rangle - g([X, Y], Z)$$
$$= (\nabla_X g)(Y, Z) - (\nabla_Y g)(X, Z) + g(T(X, Y), Z).$$

Here $g_Y = g(Y, \ ) \in \Omega^1(M)$ corresponds to $\gamma_{\tilde{Y}}$ as in the preceding corollary, and its covariant derivative by ∇_X therefore corresponds to $D_{\tilde{X}}\gamma_{\tilde{Y}}$. The last line comes from $\langle \nabla_X g_Y, Z\rangle = X(g(Y, Z)) - g(Y, \nabla_X Z)$ as in (5), the usual expression for the torsion tensor as in Proposition 2.5, and the usual expression for $\nabla_X g$ as in Proposition 2.7. □

We note, using Proposition 2.5 and the Leibniz property as in Proposition 2.7, that we may also write the cotorsion form as

$$g(T_\gamma) = \nabla \wedge g_\alpha \underset{C(M)}{\otimes} g^\alpha + g(T) - g_\alpha \wedge \nabla \underset{C(M)}{\otimes} g^\alpha = \mathrm{d}g_\alpha \underset{C(M)}{\otimes} g^\alpha - g_\alpha \wedge \nabla \underset{C(M)}{\otimes} g^\alpha. \tag{11}$$

This gives an immediate corollary:

Corollary 2.10. *In the setting of Proposition 2.9, suppose that the torsion vanishes. Then the skew-symmetrised cotorsion vanishes iff* $\mathrm{d}g = 0$, *where* d *is extended to 2-cotensors as a graded-derivation.*

Proof. We project (11) to $\Omega^3(M)$. Then the projection of $g(T_\gamma)$ is $\mathrm{d}g + g_\alpha \wedge \langle T, g^\alpha\rangle$ using Proposition 2.5. □

If we call $g(\ , T) \in \Omega^1(M) \otimes_{C(M)} \Omega^2(M)$ the *torsion form*, the corollary says that the skew-symmetrised cotorsion form and torsion form differ by $\mathrm{d}g$. For example, if the torsion vanishes and g is antisymmetric so that $\mathrm{d}g$ is the De Rahm exterior differential, the vanishing of the skew-symmetric cotorsion is the same as saying that g is a symplectic 2-form, i.e. symplectic geometry is naturally included in our generalisation. Finally, when the frame resolution bundle is trivial, γ corresponds to $f \in \Omega^1(M, V^*)$ (which we call a *V-cobein*), and $g = f \dot{\otimes}_{C(M)} e$. The vanishing of torsion and cotorsion with respect to a gauge field A is then the symmetrical condition

$$D_A e = 0, \quad D_A f = 0. \tag{12}$$

As an example, if G is any semisimple Lie group, we consider $G \times G$ as a principal G-bundle (where G acts on the right factor from the right). We take $V = \mathfrak{g}$ with the adjoint action. On G is a canonical Maurer–Cartan form with values in $\mathfrak{g}$. We take this for e and we let $f = \eta \circ e$ where η is the Killing form. This defines the usual metric on G (Riemannian in the compact case). One also has a natural gauge field A inducing a covariant derivative with vanishing torsion and cotorsion. It is given by A equal to $\frac{1}{2}$ the Maurer–Cartan form. We interpret $\mathrm{d}A + 2A \wedge A = 0$ variously as (12) and nonzero curvature. We can similarly treat homogeneous spaces G/H.

Finally, as well as suggesting such natural generalisations of conventional Riemannian geometry, the gauge field formulation also suggests more radical approaches to quantisation of the graviton. For example, one may regard the frame resolution data $(P, G, V, \theta, \gamma)$ as part of the 'background manifold' data in which our particles move, considering only the connection or gauge field A part as dynamic. There are clearly a number of interesting directions suggested by the above formulation, some of which will be explored elsewhere. The main point for our present purposes, however, is that it provides a clean 'co-ordinate free' and 'differential form' approach suitable for generalisation to quantum group noncommutative geometry, to which we now turn.

3. Quantum group Riemannian geometry

Motivated by the above reformulation of classical Riemannian geometry, we now let M be a possibly noncommutative unital algebra over a general ground field k. We use the theory of quantum principal bundles P with quantum structure group H, as introduced in [4]. Here H coacts by Δ_R on P, $M = P^H$ is the fixed point subalgebra, P is assumed to flat as an M-module and the extension is assumed to be Hopf–Galois. These definitions are somewhat like the classical ones but with arrows reversed since our spaces M, etc. are replaced by algebras playing the role of their ring of functions. We use the universal differential calculi $\Omega^1 M$, $\Omega^1 P$, etc. associated to any unital algebra. Here $\Omega^1 M \subset M \otimes M$ is the kernel of the product map and $\mathrm{d} : M \to \Omega^1 M$ is $\mathrm{d}m = 1 \otimes m - m \otimes 1$. Further, one finds [4] effectively that the role of local triviality in the theory of connections can be played by the assumption that the map $\chi : P \otimes_M P \to P \otimes H$ defined by the descent to $P \otimes_M P$ of $\tilde{\chi} = (\cdot \otimes \mathrm{id})(\mathrm{id} \otimes \Delta_R)$ is invertible. This 'Hopf–Galois' condition is familiar in Hopf algebra theory and has been considered as a natural 'topological' requirement [17] even without the differential calculus and gauge theory for this setting introduced in [4]. We assume that the quantum group H has invertible antipode.

As shown in [6] and cf. [4,7], if V is a right H-comodule with invariant 'unit' element $1 \in V$, we have an associated fibre bundle $\mathcal{E} = (P \otimes V)^H$ containing M as $M \otimes 1$. The sections of $\mathcal{E}$ are the unital left M-module maps $\mathcal{E} \to M$ covering this inclusion of M, and are in 1–1 correspondence with unital equivariant maps $V \to P$. To this theory we now add a description of $\Omega^n M$ in terms of $\mathcal{E}$. These kinds of results do not actually need a unit element in V and we drop this for the purposes of the present paper. We also recall from [4] that $\theta : V \to \Omega^1 P$ is right strongly tensorial (r.s.t.) if it is equivariant and its image lies in $P(\Omega^1 M)$.

Lemma 3.1. *Right strongly tensorial $\theta : V \to P\Omega^n M$ are in 1–1 correspondence with maps left M-module maps $\mathcal{E} \to \Omega^n M$.*

Proof. This is the analogue of Proposition 2.3 in the sense that we identify the sections of the bundle $\mathcal{E}^* = (P \otimes V^*)^H$ with equivariant maps $V^* \to P$, etc. (i.e. with elements of $\mathcal{E} = (P \otimes V)^H$). Here V^* is the right dual, i.e. has a coaction such that the evaluation map $V \otimes V^* \to \mathbb{C}$ is an intertwiner. In practice, we now proceed directly with $\mathcal{E}$ and V, without assuming V^*. Given θ, we define $\tilde{s}_\theta : P \otimes V \to P \otimes M^{\otimes n}$ by $\tilde{s}_\theta(p \otimes v) = p\theta(v)$ and verify that its restriction $s_\theta : \mathcal{E} \to \Omega^n M$ to $\mathcal{E} = (P \otimes V)^H$ indeed has its image in $\Omega^n M$. To see this, apply the coaction Δ_R to P in $P \otimes M^{\otimes n}$. Conversely, given $s : \mathcal{E} \to \Omega^n M$, let $\theta_s(v) = \chi_\alpha^{-1}(1 \otimes S^{-1}v^{(\bar{2})}) \cdot s(\chi^{-1\alpha}(1 \otimes S^{-1}v^{(\bar{2})}) \otimes v^{(\bar{1})})$ which one may verify is well defined and lies in $P\Omega^n M$. Here $\Delta_R(v) = v^{(\bar{1})} \otimes v^{(\bar{2})}$ (summation understood) is the coaction on v and $\chi^{-1} = \chi_\alpha^{-1} \otimes \chi^{-1\alpha}$ is another notation (summation over terms labelled by α understood). The proof follows exactly the same lines as the $n = 0$ case in [6,7]. □

Motivated by Corollary 2.2 we define a *frame resolution* of $\Omega^1 M$ as follows:

Definition 3.2. A frame resolution of $(M, \Omega^1 M)$ is a quantum group principal bundle $P(M, H)$ over M, a right H-comodule V and a right strongly tensorial form $\theta : V \to P\Omega^1 M$ such that the map $s_\theta : \mathcal{E} \to \Omega^1 M$ is an isomorphism.

Note that we do not fix V or H here, so this is not necessarily the frame bundle in the classical case. However, the moral of our formulation of classical Riemannian geometry in Section 2 is precisely that it does not really matter which frame resolution we take as all of them achieve the desired result that the (co)tangent bundle is expressed as an associated vector bundle. We can go on to define the 'frame bundle' as a particular frame resolution (e.g. in some sense minimal such that there is a unique torsion free connection for every metric), but we do not need to do this (it will be attempted elsewhere).

We can now proceed to conclude further isomorphisms

$$\mathrm{id} \otimes s_\theta : ((\Omega^1 M)P \otimes V)^H \cong \Omega^1 M \underset{M}{\otimes} \Omega^1 M \tag{13}$$

(i.e. with left strongly tensorial forms $V^* \to (\Omega^1 M)P$), etc.

Proposition 3.3. *Let ω be a left strong connection on P in the sense of preserving left strong tensoriality. Then the covariant derivative $D = (\mathrm{id} - \Pi_\omega)\,\mathrm{d}$ on the associated bundle $\mathcal{E}^*$ corresponds to a map $\nabla : \Omega^1 M \to \Omega^1 M \otimes_M \Omega^1 M = \Omega^2 M$ obeying the derivation property with respect to left-multiplication by M. Explicitly,*

$$\nabla = 1 \otimes \mathrm{id} - (\mathrm{id} \otimes \theta) \circ s_\theta^{-1} - s_{\theta\alpha}^{-1(\bar{1})} \cdot \omega(s_{\theta\alpha}^{-1(\bar{2})}) \cdot \theta(s_\theta^{-1\alpha}),$$

where $s_{\theta\alpha}^{-1} \otimes s_\theta^{-1\alpha}$ (summation over terms labelled by α understood) denotes the output of s_θ^{-1} after acting on $\Omega^1 M$.

Proof. By definition, a connection on P is called (left) strong if it sends pseudotensorial forms $f : V^* \to P$ (which are automatically left and right tensorial) to left strongly tensorial forms $Df : V^* \to (\Omega^1 M)P$. (The need for such a restriction is explained in [4] and it is studied in [5].) We view f more directly as an element of $\mathcal{E}$ and $Df \in ((\Omega^1 M)P \otimes V)^H$, and hence via the above isomorphisms we obtain the corresponding covariant derivative ∇ on forms as $\nabla = (\mathrm{id} \otimes s_\theta)((\mathrm{id} - \Pi_\omega)\,\mathrm{d} \otimes \mathrm{id})s_\theta^{-1}$. Putting in the form of d and $\Pi_\omega(p \otimes p') = pp'^{(\bar{1})}\omega(p'^{(\bar{2})})$ for all $p, p' \in P$ from [4] gives the explicit expression shown for ∇. From this it is immediate that the derivation property $\nabla(f.w) = f\nabla w + \mathrm{d}f \wedge w$ holds for all

$f \in M$ and $w \in \Omega^1 M$. Note that the product $\wedge$ in the universal case is the product of the adjacent copies of M. □

This constructs the covariant derivative on $\Omega^1 M$, which is the starting point of quantum group Riemannian geometry. Next we define the torsion tensor (motivated by Proposition 2.5) as $\mathrm{d} - \nabla$. We have a problem, however, that θ is right strongly tensorial not left strongly tensorial, so $D\theta$ is not strongly tensorial. This problem is resolved as follows. Note that assuming H has invertible antipode, $\bar{H} = H^{\mathrm{op}}$ (with the opposite product) is also a Hopf algebra and $\bar{P} = P$ as an algebra but with the coaction

$$\Delta_L(p) = S^{-1} p^{(\bar{2})} \otimes p^{(\bar{1})} \tag{14}$$

makes $\bar{P}$ into a left comodule algebra under $\bar{H}$. Throughout the paper, whenever we consider both left coactions and right coactions $\Delta_R p = p^{(\bar{1})} \otimes p^{(\bar{2})}$ on the same object, they will always be related like this. We define $\bar{\Pi}_\omega : \Omega^1 P \to \Omega^1 P$ by the restriction to $\Omega^1 P$ of

$$\bar{\Pi}_\omega(p \otimes p') = \omega(p^{(\tilde{1})}) p^{(\tilde{2})} p'$$

for $p, p' \in P$ and $\Delta_L p = p^{(\tilde{1})} \otimes p^{(\tilde{2})}$ is a notation (summation understood). This is exactly a left-handed version of the formula for Π_ω in [4]. We define $\bar{D} = (\mathrm{id} - \bar{\Pi}_\omega)\,\mathrm{d}$ on forms on P.

Proposition 3.4. *The following three are equivalent:*

(i) *ω is a left strong connection on P,*

(ii) $$(\Delta_L \otimes \mathrm{id}) \circ \omega = \mathrm{id} \otimes 1 \otimes 1 - 1 \otimes 1 \otimes 1\epsilon + (\mathrm{id} \otimes \omega) \circ \Delta, \tag{15}$$

(iii) $$(\mathrm{id} \otimes \Delta_R)\omega = 1 \otimes 1 \otimes \mathrm{id} - 1 \otimes 1 \otimes 1\epsilon + (\omega \otimes \mathrm{id}) \circ \Delta. \tag{16}$$

Moreover, in this case $\bar{D}$ preserves right strongly tensorial form.

Proof. By definition, ω is left strong if D preserves left strongly tensorial forms. As explained in [6], this is equivalent to sending the identity map $P \to P$ regarded as a (left) strongly tensorial form to a left strongly tensorial form, which is the condition studied in [5]. The explicit form of this with the universal calculus in [6] can be massaged further (using the comodule algebra property of Δ_R) as

$$\begin{aligned} &p^{(\bar{1})}\omega_\alpha(p^{(\bar{2})}{}_{(2)})^{(\bar{1})} \otimes p^{(\bar{2})}{}_{(1)}\omega_\alpha(p^{(\bar{2})}{}_{(2)})^{(\bar{2})} \otimes \omega^\alpha(p^{(\bar{2})}{}_{(2)}) \\ &\quad = p \otimes 1 \otimes 1 - \Delta_R p \otimes 1 + p^{(\bar{1})}\omega_\alpha(p^{(\bar{2})}) \otimes 1 \otimes \omega^\alpha(p^{(\bar{2})}), \end{aligned}$$

where $\omega_\alpha \otimes \omega^\alpha = \omega$ is a notation. Multiplying all expressions here from the left by $p' \in P$, we note that they can all be factored through the map $\chi(p' \otimes_M p) = p' \Delta_R(p)$, which is invertible by the Hopf–Galois condition. So, applying to $\chi^{-1}(1 \otimes h)$ for $h \in H$, we obtain the simpler but equivalent condition

$$\begin{aligned} &\omega_\alpha(h_{(2)})^{(\bar{1})} \otimes h_{(1)}\omega_\alpha(h_{(2)})^{(\bar{2})} \otimes \omega^\alpha(h_{(2)}) \\ &\quad = \epsilon(h)1 \otimes 1 \otimes 1 - 1 \otimes h \otimes 1 + \omega_\alpha(h) \otimes 1 \otimes \omega^\alpha(h). \end{aligned} \tag{17}$$

Finally, we apply this equation to $h_{(2)}$ of $Sh_{(1)} \otimes h_{(2)}$ and multiply the free $Sh_{(1)}$ in from the left. This gives the simpler equation

$$(\Delta_R \otimes \mathrm{id})\omega(h) = 1 \otimes Sh \otimes 1 - \epsilon(h)1 \otimes 1 \otimes 1 + \omega_\alpha(h_{(2)}) \otimes Sh_{(1)} \otimes \omega^\alpha(h_{(2)}).$$

This is the condition stated in terms of Δ_L and is equivalent so long as S is invertible. On the other hand, ω as a connection is Ad invariant. Using this fact, we may convert the coaction on the first output of ω on the left-hand side of (17) to a coaction on the second output of ω:

$$\begin{aligned}&\omega_\alpha(h_{(1)})\otimes\omega^\alpha(h_{(2)})^{(\tilde 1)}\cdot_{\mathrm{op}} h_{(2)}\otimes\omega^\alpha(h_{(2)})^{(\tilde 2)}\\ &\quad=\epsilon(h)1\otimes1\otimes1-1\otimes h\otimes1+\omega_\alpha(h)\otimes1\otimes\omega^\alpha(h)\end{aligned}\tag{18}$$

as equivalent to (17) by Ad-invariance of ω. This is the same condition as (17) with left–right reversed and H replaced by $\bar H$. We can also apply it to $h_{(1)}$ of $h_{(1)}\otimes h_{(2)}$ and multiplying in the $h_{(2)}$ gives us the equivalent expression (16).

Moreover, if $\bar P(\bar H, M, \Delta_L)$ is a left-handed quantum principal bundle, then we can immediately conclude the result by left–right symmetry: view a right strongly tensorial form on P equivalently as a strongly tensorial form on $\bar P$: the condition that such forms are preserved under $\bar D$ is exactly the Δ_L version of (17), which have seen is equivalent to ω left strong. More generally, without assuming $\bar P$ is Hopf–Galois we define $\bar D$ directly by the same formulae as in the Hopf–Galois case and verify that it preserves right strongly tensorial forms. □

The proposition suggests the definition of a quantum bundle as *bicovariant* if both P, $\bar P$ obey the Hopf–Galois condition. In this case, the above lemma asserts that a connection is (left) strong on P *iff* it is (right) strong on $\bar P$. We do not need to assume this at the moment since we are interested only in working on P.

Proposition 3.5. *The torsion tensor $T:\Omega^1M\to\Omega^1M\otimes_M\Omega^1M$ defined as $T=\mathrm{d}-\nabla$ corresponds under the isomorphism in Lemma 3.1 to the right strongly tensorial form $\bar D\theta: V\to P\Omega^2M$. Explicitly,*

$$\bar D\theta=1\otimes\theta-\theta_\alpha\otimes1\otimes\theta^\alpha+\theta\otimes1+\cdot\circ(\omega\otimes\theta)\circ\Delta_L,$$

where $\theta=\theta_\alpha\otimes\theta^\alpha$ (say) and $\cdot$ multiplies the adjacent copies of P between ω and θ, and Δ_L is related as in (14) to the original right coaction on V.

Proof. The exterior derivative on $\Omega^1M\subset M\otimes M$ is the restriction of $\mathrm{d}(m\otimes n)=1\otimes m\otimes n-m\otimes1\otimes n+m\otimes n\otimes1$. Subtracting $\nabla(m\otimes n)$ given in Proposition 3.5 and precomposing all terms with s_θ gives $(\mathrm{d}-\nabla)\circ s_\theta:\mathcal{E}\to\Omega^2M$. By Lemma 3.1 it has the form $\cdot\circ(\mathrm{id}\otimes Y)$ for some 1-form $Y:V\to P\Omega^2M$. Computing Y gives the same expression as that stated for $\bar D\theta$. On the other hand, this is what one obtains as $(\mathrm{id}-\bar\Pi_\omega)\,\mathrm{d}\theta$ by similar (but left-handed) computations to those in [4]. □

Clearly, a torsion free connection with respect to a given frame resolution is one where $\bar D\theta=0$. Let us note that in the case of the universal calculus where $\Omega^2M=\Omega^1M\otimes_M\Omega^1M$, this condition is much stronger than it would be classically or with nonuniversal calculi. Since ∇ coincides with $\nabla\wedge$, we see that the vanishing of torsion implies a unique covariant derivative, namely $\nabla=\mathrm{d}$. One may also see by applying s_θ^{-1} to the explicit formula for $\bar D\theta$ that ω composed with the left-handed coaction $\Delta_L: V\to H\otimes V$ is fully determined by

$\bar{D}\theta$. One has uniqueness of the composition for prescribed torsion. Moreover, by considering $\bar{D}^2\theta$, one finds that the curvature of such a ω must vanish. These are intrinsic limitations of the universal calculus.

Next, we can go on and define metric-compatibility as $\nabla g = 0$, where ∇ is suitably extended to $g \in \Omega^1 M \otimes_M \Omega^1 M$, for example, as a derivation if we follow the line of Proposition 2.7. On the other hand, Corollary 2.8 suggests a different formulation as follows. We consider g equivalently as $\gamma \in ((\Omega^1 M)P \otimes V)^H$ via the isomorphisms above, nondegenerate in the sense of providing a dual (and left-handed) frame resolution via V^*. Thus,

$$g = \gamma(f^a)\theta(e_a) \in \Omega^1 M \underset{M}{\otimes} \Omega^1 M \subset M \otimes M \otimes M, \tag{19}$$

where we assume the existence of the canonical element or coevaluation $f^a \otimes e_a = \mathrm{coev} \in V^* \otimes V$ for the duality pairing of V^* with V (here $\{e_a\}$ is a basis of V and $\{f^a\}$ a dual basis.) When V is infinite-dimensional one can consider (19) formally as a power series or alternatively one can replace γ by a *left-handed* frame resolution $\theta_L : V \to (\Omega^1 M)P$ (i.e. a left strongly tensorial form such that $s_{\theta_L}(v \otimes p) = \theta_L(v)p$ induces an isomorphism $(V \otimes M)^H \cong \Omega^1 M)$ and an invariant element $\eta = \eta^{(1)} \otimes \eta^{(2)} \in V \otimes V$. Then

$$g = \theta_L(\eta^{(1)})\theta(\eta^{(2)}) \in \Omega^1 M \underset{M}{\otimes} \Omega^1 M \subset M \otimes M \otimes M. \tag{20}$$

In practice η will still tend to be in some completed tensor product when we work with the universal calculus, and it should be nondegenerate in some sense. As soon as we pass to nonuniversal calculi, V will tend to be finite-dimensional and these subtleties will not arise. Therefore, for simplicity, we stress the version with γ. Finally, we have seen that vanishing of the torsion and cotorsion classically means metric compatibility in a skew symmetrised or 'differential form' sense. However, quantum differential forms in the universal calculus are not skew symmetrised so at least in this setting we can reasonably take $D\gamma = 0$ and $\bar{D}\theta = 0$ as the correct conditions for 'torsion free and metric compatible'.

Proposition 3.6. *We define the cotorsion form $\Gamma \in \Omega^2 M \otimes_M \Omega^1 M$ as corresponding to $D\gamma$ under* $\mathrm{id} \otimes s_\theta$. *It coincides with $(T_\gamma \otimes \mathrm{id})g$ where T_γ is the right M-module map corresponding in the dual frame resolution to the torsion of γ. Moreover,*

$$\Gamma = \mathrm{d}g + (\mathrm{id} \otimes T)g.$$

Proof. The explicit computations are similar (with a left–right reversal) to those already made for $\bar{D}\theta$, etc. so we omit them. Comparing the result of $\Gamma = (T_\gamma \otimes \mathrm{id})g$ with that of $(\mathrm{id} \otimes T)g$, we find that they differ by $\mathrm{d}g$. □

Here $(\mathrm{id} \otimes T)g$ is the torsion form (i.e the torsion tensor viewed in $\Omega^1 M \otimes_M \Omega^2 M$ via the quantum metric). So the proposition says that the cotorsion minus the torsion is $\mathrm{d}g$. Since the universal differential calculus has trivial cohomology, we see that for a torsion free connection, a metric γ has zero cotorsion *iff* the corresponding g is exact.

As a third formulation of the quantum metric, one should be able to identify $\Omega^1 M \otimes_M \Omega^1 M$ with $(P \otimes (V \otimes V))^H$, i.e. with tensorial $V \otimes V$ 0-forms. This is a different quantum generalisation of ∇ from the derivation property, as we have seen in detail in the proof of Proposition 2.7 in Section 2. This third approach, however, will be considered elsewhere because it appears to need nonuniversal calculi for its proper formulation. The main results above clearly do extend in principle to nonuniversal calculi $(M, \Omega^1(M))$. Thus, a frame resolution means a choice of quantum principal bundle with nonuniversal calculus $(P, \Omega^1(P), H, \Omega^1(H), V, \theta)$ as in [4,8] and a right strongly tensorial form $\theta : V \to \Omega^1(P)$ such that the induced map $s_\theta : \mathcal{E} \to \Omega^1(M)$ is an isomorphism. Similarly, we may introduce a metric as $\gamma : V^* \to \Omega^1(P)$ such that the induced s_γ is an isomorphism. The underlying theory of associated bundles with nonuniversal calculi needs to be developed first in order to proceed further.

Finally, we recall that a quantum principal bundle (P, H) is called trivial [4] if there is a convolution-invertible unital linear map $\Phi : H \to P$ which intertwines the right regular coaction of H on itself and Δ_R on P. In terms of extension theory of algebras, one says that the extension is cleft, and one knows that P is then a cocycle cross product algebra. A strong connection ω in this case is equivalent to a 'gauge field' $A : H \to \Omega^1 M$ such that $A(1) = 0$. We have [4]:

$$\omega = \Phi^{-1} * A * \Phi + \Phi^{-1} * \mathrm{d}\Phi, \tag{21}$$

where $*$ is the convolution product $x * y = \cdot \circ (x \otimes y) \circ \Delta$ for maps x, y from H. From [4], we also know that left strongly tensorial forms such as (in our present case) $\gamma : V^* \to (\Omega^1 M)P$ on a trivial bundle are in 1–1 correspondence with 'matter fields' $f : V^* \to \Omega^1 M$ via $\gamma = f *_R \Phi$. (Similarly, $\theta_L = e_L *_R \Phi$ for some $e : V \to \Omega^1 M$ in the alternative formulation of the quantum metric.) Here we extend the $*$ notation to the convolution right action with respect to Δ_R in place of Δ. In a similar way, one finds that right strongly tensorial forms such as (in our case) $\theta : V \to P\Omega^1 M$ are in 1–1 correspondence with 'matter fields' $e : V \to \Omega^1 M$ via $\theta = \Phi^{-1} *_L e$, where we now also use $*_L$ for the convolution left action with respect to Δ_L. Moreover, that the maps s_θ and s_γ are invertible correspond, respectively, to invertibility of the maps

$$\begin{aligned} &s_e : M \otimes V \to \Omega^1 M, \qquad s_e(m \otimes v) = m e(v),\\ &s_f : V^* \otimes M \to \Omega^1 M, \qquad s_f(w \otimes m) = f(w) m \end{aligned} \tag{22}$$

for $m \in M$, $v \in V$, $w \in V^*$. The first case follows easily from the description in [4] of the associated bundle $\mathcal{E}$ in the trivial bundle case as $M \otimes V \cong \mathcal{E}$, combined with Lemma 3.1, while the second is similar. We similarly need s_{e_L} an isomorphism in that setting. We call these particular 'matter fields' e and f the *quantum V-bein or V-cobein*, respectively, and e_L a left-handed quantum V-bein. They are global parallelisations of $\Omega^1 M$ as a basis of 1-forms over M acting either from the left or from the right.

Proposition 3.7. *For a trivial quantum principal bundle frame resolution in terms of gauge fields and V-(co)beins, the covariant derivative is*

$$\nabla = 1 \otimes \mathrm{id} - (\mathrm{id} \otimes e) s_e^{-1} - s_{e\alpha}^{-1} \cdot (A *_L e)(s_e^{-1\alpha}),$$

where $s_{e\alpha}^{-1} \otimes s_e^{-1\alpha}$ denotes the output of s_e^{-1} acting on $\Omega^1 M$. The torsion and cotorsion correspond to

$$\bar{D}_A e = \mathrm{d}e + A *_L e, \qquad D_A f = \mathrm{d}f + f *_R A$$

and the metric has the form $g = f(f^a)e(e_a)$, *where* $\{e_a\}$ *is a basis of* V *and* $\{f^a\}$ *is a dual basis.*

Proof. This follows from Proposition 2.3 and the form of s_e^{-1} deduced from (22), and the isomorphism $M \otimes V \cong \mathcal{E}$ from [4]. The torsion tensors and cotorsion tensor correspond to $\bar{D}\theta = \Phi^{-1} *_L \bar{D}_A e, D\gamma = (D_A f) *_R \Phi$, for some $\bar{D}_A e : V \to \Omega^2 M$, $D_A f : V^* \to \Omega^2 M$, since they are again right and left strongly tensorial. One immediately finds them as shown. One similarly has $D_A e_L = \mathrm{d}e_L + e_L *_R A$ in that setting for the quantum metric. □

Note that the requirements for s_e, s_f (or s_{e_L}) to be isomorphisms are the same as saying that (M, k, V, e) and (M, k, V^*, f) (or (M, k, V, e_L)) are frame resolutions with trivial quantum group $H = k$ and $P = M$. So M has a trivial principal bundle frame resolution and/or dual frame resolution *iff* it has a trivial one with $H = k$. On the other hand, extending P to some larger trivial frame bundle with nontrivial structure group H allows for a larger range of covariant derivatives induced by different gauge fields A. Moreover, this is also the 'local picture' when a nontrivial bundle is glued by patching together trivial bundles as in [4].

4. Basic constructions: q-homogeneous spaces and bosonisation

In this section we show how the formalism above includes various well-known quantum spaces arising in the theory of quantum groups and braided groups. We provide some basic general classes of examples as well as concrete cases such as the quantum sphere S_q^2 and the quantum planes $\mathbb{R}_q^n$. We mainly establish the existence of the frame resolution and the general form of the covariant derivative ∇ for our examples, and in some cases we obtain a quantum metric g.

We start with the simplest of all examples, namely $M = H$ a Hopf algebra. As we might expect, its universal differential calculus is 'parallelisable' in the sense that it can be resolved with trivial quantum group in the frame resolution.

Proposition 4.1. *Let* $M = H$, *a Hopf algebra. Then* $(H, \Omega^1 H)$ *has a quantum frame resolution with* $P = H$ *and structure quantum group* k, *the ground field, and*

$$V = \ker \epsilon \subset H, \qquad \theta(v) = Sv_{(1)} \otimes v_{(2)}.$$

The covariant derivative and torsion are

$$\nabla(h \otimes g) = 1 \otimes h \otimes g - hg_{(1)} \otimes Sg_{(2)} \otimes g_{(3)},$$
$$T(h \otimes g) = h \otimes g \otimes 1 - h \otimes 1 \otimes g + hg_{(1)} \otimes Sg_{(2)} \otimes g_{(3)}$$

extended linearly and restricted to $\Omega^1 H$.

Proof. Here $\mathcal{E} = H \otimes \ker \epsilon$ and $s_\theta(h \otimes v) = hSv_{(1)} \otimes v_{(2)}$ is an isomorphism $s_\theta : \mathcal{E} \to \Omega^1 H$. This map is, in fact, the inverse of the well-known isomorphism $\Omega^1 H \to H \otimes \ker \epsilon$ given by $h \otimes g \to hg_{(1)} \otimes g_{(2)}$. Hence this choice of P, V, θ indeed provides a frame resolution of $H, \Omega^1 H$. On the other hand, $\ker \epsilon \subset k$ is 0 so only $\omega = 0$ is possible. Then ∇, T necessarily have the form stated. □

This provides a 'quantum geometrical' picture of the isomorphism $\Omega^1 H \cong H \otimes \ker \epsilon$ playing a fundamental role in the theory of differential calculi on quantum groups. Similarly, any left-covariant $\Omega^1(H)$ has the form $H \otimes \ker \epsilon / Q \cong \Omega^1(H)$ where Q is a right ideal in $\ker \epsilon$ [18]. One can view this, as above, as coming from a frame resolution where $V = \ker \epsilon / Q$ and θ is inherited from the formula above.

The trivial frame resolution here induces only one covariant derivative. On the other hand, we can view θ as a quantum V-bein as part of any trivial quantum group principal bundle with quantum group H' coacting on V, giving a larger range of induced covariant derivatives according to gauge fields A (see Proposition 3.7). Moreover, by an evident left–right symmetry, we also have a left frame resolution

$$\theta_L(v) = v_{(1)} \otimes Sv_{(2)} \tag{23}$$

and hence a metric if we are given a nondegenerate $\eta \in V \otimes V$. In the finite-dimensional case when η is viewed as map $V^* \to V$ by $\eta(w) = \eta(\ , w)$, we also have a dual frame resolution $\gamma(w) = \eta(w)_{(1)} \otimes S\eta(w)_{(2)}$. The corresponding quantum metric in either case is

$$g = \eta^{(1)}{}_{(1)} \otimes S(\eta^{(2)}{}_{(1)}\eta^{(1)}{}_{(2)}) \otimes \eta^{(2)}{}_{(2)}, \tag{24}$$

where $\eta = \eta^{(1)} \otimes \eta^{(2)}$. If η is H'-invariant, then we can view γ as a quantum V-cobein on the trivial quantum principal bundle with structure quantum group H'.

Proposition 4.2. *Let H be a Hopf algebra. Then $(H, \Omega^1 H)$ has a frame resolution by $P = H \otimes H$, structure quantum group H, quantum V-bein and covariant derivative*

$$\begin{aligned} &V = \ker \epsilon, \quad \Delta_L = \Delta - \mathrm{id} \otimes 1, \quad e(v) = Sv_{(1)} \otimes v_{(2)} \\ &\nabla(h \otimes g) = 1 \otimes h \otimes g - hg_{(1)} \otimes Sg_{(2)} \otimes g_{(3)} \\ &\qquad - hg_{(1)}A(g_{(2)})Sg_{(3)} \otimes g_{(4)} + hg_{(1)}A(g_{(2)}) \otimes 1. \end{aligned}$$

There is a unique gauge field $A(h) = Sh_{(1)} \otimes h_{(2)} - \epsilon(h)1 \otimes 1$ with zero torsion. For any $\Lambda \in H$,

$$\eta = S\Lambda_{(1)} \otimes \Lambda_{(2)} - S\Lambda \otimes 1 - 1 \otimes \Lambda + \epsilon(\Lambda)1 \otimes 1 \in V \otimes V$$

is H-invariant, and when nondegenerate it defines a quantum metric

$$\begin{aligned} g = {} & S\Lambda_{(2)} \otimes S^2\Lambda_{(1)}S\Lambda_{(3)} \otimes \Lambda_{(4)} - S\Lambda_{(2)} \otimes S^2\Lambda_{(1)} \otimes 1 \\ & -1 \otimes S\Lambda_{(1)} \otimes \Lambda_{(2)} + \epsilon(\Lambda)1 \otimes 1 \otimes 1. \end{aligned}$$

There is a unique gauge field $A(h) = h_{(1)} \otimes Sh_{(2)} - \epsilon(h) 1 \otimes 1$ *with zero cotorsion. Both gauge fields have zero curvature.*

Proof. We use Proposition 3.7 as explained above, regarding $P = H \otimes H$ as a trivial bundle with structure group H and $\Delta_R(h \otimes g) = h \otimes g_{(1)} \otimes g_{(2)}$, and regarding θ, θ_L in Proposition 4.1 as V-bein and left V-bein. We equip V with the coaction Δ_L as stated, and easily verify that $\eta \in V \otimes V$ is invariant under the tensor product coaction. Also, both gauge fields obey $\mathrm{d}A + A * A = 0$, which we can interpret as $\bar{D}_A e = 0$ in the first case and $D_A e_L = 0$ in the second. By applying s_e and s_{e_L} to these equations one knows that A composed with Δ_L or Δ_R on V is fully determined. By applying ϵ, this determines A in the two cases uniquely. In the finite-dimensional case it makes sense to require nondegeneracy as an isomorphism $\eta : V^* \to V$ or equivalently as a map $V^* \otimes V^* \to k$. Equivalently, we identify $V^* = \ker \epsilon \subset H^*$ and require that $\langle \Lambda, (Sw)x \rangle$ is nondegenerate as a bilinear form on $w, x \in V^*$. □

The nondegeneracy condition here holds, for example, when H is finite-dimensional and Λ is a normalised integral in H and Λ^* a normalised integral in H^* such that $\langle \Lambda^*, \Lambda \rangle$ is invertible. For then $\eta(w) = S\Lambda_{(1)} \langle \Lambda_{(2)}, w \rangle - 1 \langle \Lambda, w \rangle$ is the Fourier transform on H^* (restricted to $\ker \epsilon$ and projected to $\ker \epsilon$) and is invertible, cf. [3, Corollary 1.5.6]. For example, we may certainly take the functions $H = \mathbb{C}(G)$ on a finite group G and Λ the Kronecker δ-function at the identity.

4.1. Quantum principal homogeneous spaces

A quantum homogeneous principal bundle is [4] a Hopf algebra surjection $\pi : P \to H$ such that $\Delta_R = (\mathrm{id} \otimes \pi) \circ \Delta$ makes P a quantum principal bundle over $M = P^H$. A sufficient condition is that the product map $\ker \epsilon|_M \otimes P \to \ker \pi$ is a surjection [4]. M is called a principal quantum homogeneous space, cf. [17]. A linear splitting $i : H \to P$ of π such that $(i \otimes \mathrm{id})\mathrm{Ad} = (\mathrm{id} \otimes \pi)\mathrm{Ad} \circ i$ defines a connection $\omega(h) = (Si(h)_{(1)})\, \mathrm{d}i(h)_{(2)}$, see [4], called the *canonical connection* associated to a splitting. Necessary and sufficient conditions for the canonical connection to be strong are in [14] and amount to i a unital bicovariant splitting.

Proposition 4.3. *Let* M *be a quantum principal homogeneous space associated to* $\pi : P \to H$. *Then* $(M, \Omega^1 M)$ *has a quantum frame resolution* (P, H, V, θ), *where*

$$V = \ker \epsilon \cap M, \qquad \theta(v) = Sv_{(1)} \otimes v_{(2)}, \qquad \Delta_R(v) = v_{(2)} \otimes \pi(Sv_{(1)}).$$

Let i *be a bicovariant unital splitting. Then the associated canonical covariant derivative is the restriction to* $\Omega^1 M$ *of*

$$\nabla(m \otimes n) = 1 \otimes m \otimes n - m_{(1)} n_{(1)} Si \circ \pi(m_{(2)} n_{(2)})_{(1)} \otimes i \circ \pi(m_{(2)} n_{(2)})_{(2)} Sn_{(3)} \otimes n_{(4)}.$$

Proof. We first check that $\theta(v) \in P \otimes M$ by computing $(\mathrm{id} \otimes \Delta_R)\theta(v) = Sv_{(1)} \otimes v_{(2)} \otimes \pi(v_{(3)}) = Sv_{(1)(1)} \otimes v_{(1)(2)} \otimes \pi(v_{(2)}) = Sv_{(1)} \otimes v_{(2)} \otimes 1 = \theta(v) \otimes 1$, using coassociativity and that $V \subset M$. We also verify that Δ_R as stated makes V a right comodule. Note that this is such that the corresponding left-handed coaction as in Section 3 is $\Delta_L(v) = \pi(v_{(1)}) \otimes v_{(2)}$. This makes it clear that Δ_R is indeed a coaction (since Δ_L clearly is) and that $\Delta_R(v) \in V \otimes H$. Indeed, clearly $\Delta_L(v) \in H \otimes M$ by the same argument as for θ_V, and $(\mathrm{id} \otimes \epsilon)\Delta_L(v) = \pi(v) = \epsilon(v_{(1)})\pi(v_{(2)}) = \epsilon(v) \otimes 1 = 0$ since $v \in M$ and then $v \in \ker \epsilon$. We similarly verify that θ is an intertwiner. Putting the output of Δ_R to the far right, we have $(\Delta_R \otimes \mathrm{id})\theta(v) = (Sv_{(1)})_{(1)} \otimes v_{(2)} \otimes \pi((Sv_{(1)})_{(2)}) = Sv_{(2)} \otimes v_{(3)} \otimes S\pi(v_{(1)}) = Sv^{(\bar{1})}{}_{(1)} \otimes v^{(\bar{1})}{}_{(2)} \otimes v^{(\bar{2})} = \theta(v^{(\bar{1})}) \otimes v^{(\bar{2})}$ by coassociativity and the form of $\Delta_R v = v^{(\bar{1})} \otimes v^{(\bar{2})}$. Hence the various maps are defined as stated. By Lemma 3.1 we have an induced $s_\theta(p \otimes v) = pSv_{(1)} \otimes v_{(2)}$ and we verify that it is an isomorphism $s_\theta : P \otimes V \cong \Omega^1 M$. Indeed, we define the inverse as the restriction to $\Omega^1 M$ of $s_\theta^{-1}(m \otimes n) = mn_{(1)} \otimes n_{(2)}$. This has its right-hand output in M by the same coassociativity argument as above. Moreover, $(\mathrm{id} \otimes \epsilon)s_\theta^{-1}(m \otimes n) = mn$ so $\Omega^1 M$ maps to $P \otimes V$ as required. That the two maps s_θ and s_θ^{-1} are mutually inverse is the same elementary computation as in Proposition 4.1. Indeed, these maps are restrictions of the corresponding maps for P as a Hopf algebra with its trivial frame resolution. Finally, putting in the form of ω into Proposition 3.3 immediately gives ∇ as shown. We note that this simplifies slightly on exact forms, as

$$\begin{aligned}\nabla(\mathrm{d}m) = {} & 1 \otimes 1 \otimes m - 1 \otimes m \otimes 1 + m \otimes 1 \otimes 1 \\ & - m_{(1)}Si \circ \pi(m_{(2)})_{(1)} \otimes i \circ \pi(m_{(2)})_{(2)}Sm_{(3)} \otimes m_{(4)} \end{aligned} \tag{25}$$

for all $m \in M$. Also, $T(\mathrm{d}m) = -\nabla(\mathrm{d}m)$. Since ∇ is left derivation and the torsion tensor is a left-module map, they are fully defined by their values on exact forms. □

The most well-known nontrivial example of a principal quantum homogeneous space is the quantum sphere $M = S_q^2$, where $P = SO_q(3)$ as the even subalgebra of $SU_q(2)$ with usual generators $\alpha, \beta, \gamma, \delta$, and $H = k[z, z^{-1}]$ with projection and induced Δ_R

$$\pi\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix}, \qquad \Delta_R\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha \otimes z^{1/2} & \beta \otimes z^{-1/2} \\ \gamma \otimes z^{1/2} & \delta \otimes z^{-1/2} \end{pmatrix},$$

restricted to $SO_q(3)$. Similarly to [4], one can take $i(z^n) = \alpha^{2n}$ and $i(z^{-n}) = \delta^{2n}$ and verify that one has a strong canonical connection (the charge 2 monopole). One may also take a slightly more complicated i in trivial bundle 'patches' if one wants closer contact with the classical formulae [4]. The S_q^2 is the subalgebra generated by $1, b_- = \alpha\beta, b_+ = \gamma\delta, b_3 = \alpha\delta$ and is an example of the family in [19]. One may then compute the covariant derivatives $\nabla(\mathrm{d}b_3), \nabla(\mathrm{d}b_\pm)$ and verify that they are nonzero.

4.2. Quantum planes and other braided groups

Other natural 'quantum geometries' (in the sense of being associated naturally with quantum group symmetries) are braided groups B. These (for our purposes here) are covariant objects under some background strict quantum group H, which, like the $\mathbb{Z}_2$ of supersymmetry, induces 'braid statistics' on B. Basic examples (all due to the author) are quantum planes, q-Minkowski space and versions BG_q for all the standard quantum groups, see [2,3]. In this section we consider quantum group Riemannian geometry on such objects, i.e. we take $M = B$.

First of all, the braided version of Proposition 4.1 follows in just the same way: $V = \ker \underline{\epsilon} \subset B$ and $\theta(v) = \underline{S}v_{\underline{(1)}} \otimes v_{\underline{(2)}} \in \Omega^1 B$, where $\underline{\epsilon}$, $\underline{S}$, $\underline{\Delta}$ are the braided group counit, antipode and coproduct (the underlines are to remind us that braided groups are not quantum groups in the usual way, having braid statistics). Thus every braided group B is 'parallelisable' with frame resolution by trivial quantum (or braided) group k. We similarly have $\gamma = \theta_L \circ \eta$ given any isomorphism $\eta : V^* \to V$, and $\theta_L(v) = v_{\underline{(1)}} \otimes \underline{S}v_{\underline{(2)}}$ as a left-handed version of θ. The corresponding quantum metric is shown in Fig. 6(d) in Appendix A.

On the other hand, we know from bosonisation theory [3, Theorem 9.4.12] that every braided group has an equivalent quantum group $B{>\!\!\!\triangleleft} H$ given by adjoining the background covariance quantum group. So we can view the trivial resolution instead as a quantum V-bein.

Proposition 4.4. *Let $M = B$ be a braided group covariant under a dual quasitriangular Hopf algebra $(H, \mathcal{R})$ by left coaction $\Delta_L(b) = b^{(\bar{1})} \otimes b^{(\bar{2})}$. Then $(B, \Omega^1 B)$ has a frame resolution with trivial bundle $P = B{>\!\!\!\triangleleft} H$, $V = \ker \underline{\epsilon} \subset B$ and quantum V-bein $e(v) = \underline{S}v_{\underline{(1)}} \otimes v_{\underline{(2)}}$. The covariant derivative induced by a gauge field A is*

$$\nabla(\mathrm{d}b) = 1 \otimes 1 \otimes b - 1 \otimes b \otimes 1 + b \otimes 1 \otimes 1 - b_{(1)} \otimes Sb_{(2)} \otimes b_{(3)}$$
$$+ b_{\underline{(1)}} A({b_{\underline{(2)}}}^{(\bar{1})} {b_{\underline{(3)}}}^{(\bar{1})}) \underline{S}{b_{\underline{(2)}}}^{(\bar{2})} \otimes {b_{\underline{(3)}}}^{(\bar{2})}.$$

Moreover, if $\eta \in V \otimes V$ is nondegenerate and H-covariant, then

$$g = {\eta^{(1)}}_{\underline{(1)}} \otimes (\underline{S}{\eta^{(1)}}_{\underline{(2)}}) \underline{S}{\eta^{(2)}}_{\underline{(1)}} \otimes {\eta^{(2)}}_{\underline{(2)}}$$

is a quantum metric.

Proof. (cf. [4]), We view $B{>\!\!\!\triangleleft} H$ as a quantum principal bundle with trivialisation $\Phi(h) = 1 \otimes h$ and $\Phi^{-1} = S \circ \Phi$. The quantum V-bein e induces a canonical form

$$\theta(v) = (\Phi^{-1} *_L e)(v) = (1 \otimes Sv^{(\bar{1})}) \cdot (\underline{S}{v^{(\bar{2})}}_{\underline{(1)}} \otimes 1) \otimes ({v^{(\bar{2})}}_{\underline{(2)}} \otimes 1),$$

where the product $\cdot$ is in $B{>\!\!\!\triangleleft} H$. In our case $(1 \otimes h) \cdot (b \otimes 1) = h_{(1)} \triangleright b \otimes h_{(2)} = b^{(\bar{2})} \otimes h_{(2)} \mathcal{R}(b^{(\bar{1})} \otimes h_{(1)})$ depends on the quasitriangular structure if one wants to compute θ explicitly. Similarly, we have a left-handed quantum V-bein $e_L(v) = v_{\underline{(1)}} \otimes \underline{S}v_{\underline{(2)}}$ which combined with η gives the metric as shown. One may view η as a map $V^* \to V$ by $\eta(w) = (\mathrm{id} \otimes w)\eta$ and consider that $\gamma = \theta_L \circ \eta$ is the quantum cobein, where

$$\theta_L(v) = (e_L *_R \Phi)(v) = ({v^{(\bar{2})}}_{\underline{(1)}} \otimes 1) \otimes (\underline{S}{v^{(\bar{2})}}_{\underline{(2)}} \otimes Sv^{(\bar{1})}).$$

Here $\Delta_R(v) = v^{(\bar{2})} \otimes Sv^{(\bar{1})}$ is the right action on V corresponding to Δ_L and invariance of η ensures that η as a map is covariant. $\square$

As the simplest example, we consider $M = B = k[x]$ the 'braided line' [20,21], with background quantum group $H = k[\varsigma, \varsigma^{-1}]$ and $\mathcal{R}(\varsigma^m \otimes \varsigma^n) = q^{mn}$. The covariance under the coaction is $\Delta_L(x^m) = \varsigma^m \otimes x^m$ and corresponds to the $\mathbb{Z}$-grading of $k[x]$ by degree. The bosonisation $P = k[x] >\!\!\triangleleft k[\varsigma, \varsigma^{-1}]$ is the quantum plane generated by x, ς with ς^{-1} adjoined (or 'quantum cylinder'). A gauge field is any $A : k[\varsigma, \varsigma^{-1}] \to \Omega^1 k[x]$ such that $A(1) = 0$. This means a collection of 1-forms $A(\varsigma^m) \in \Omega^1 k[x]$ for $m \neq 0$. The computations are easily made from the preceding proposition. For example,

$$e(x^m) = \sum_{r=0}^{m} \begin{bmatrix} m \\ r \end{bmatrix}_q (-1)^r x^{r(r-1)/2} \otimes x^{m-r},$$

where $[{m \atop r}]_q$ are the q-binomial coefficients. One may similarly compute the covariant derivative using the q-trinomial coefficients for the coefficients of $(\mathrm{id} \otimes \underline{\Delta}) \circ \underline{\Delta} x^m$. For the lowest generators, one has

$$\nabla \mathrm{d}x = A(\varsigma)\, \mathrm{d}x,$$
$$\nabla \mathrm{d}x^2 = A(\varsigma^2)\, \mathrm{d}x^2 + (1+q)((\mathrm{d}x)\, \mathrm{d}x + x A(\varsigma)\, \mathrm{d}x - A(\varsigma^2) x\, \mathrm{d}x).$$

The sl_n quantum-braided planes $M = \mathbb{R}_q^n$ are similarly braided groups [20] covariant under $\widetilde{SL_q(n)} = GL_q(n)$. Their bosonisations $\mathbb{R}_q^n >\!\!\triangleleft GL_q(n)$ are therefore now to be regarded via Proposition 4.4 as the *linear frame bundles* of the quantum planes. In general, and unlike the classical situation, one has different q-deformed versions of $\mathbb{R}_q^n$ of various covariance types, for example, associated to covariance under all the dilatonic extension $\widetilde{G_q}$ of the standard matrix quantum groups G_q. The general construction is provided [20] by the theory of linear braided groups $B = V_L(R', R)$ where R', R are certain 'R-matrix' data and covariance is under the dilatonic extension of a quantum group obtained from the quantum matrix bialgebras [22] $A(R)$. The bosonisation of these quantum braided planes provides the construction of inhomogeneous quantum groups $\mathbb{R}_q^n >\!\!\triangleleft \widetilde{G_q}$, etc. in [20], which we understand now as frame resolutions by $H = \widetilde{G_q}$ of these various quantum planes.

The obvious case, using the so_n series R-matrix, is the quantum Poincaré algebra $\mathbb{R}_q^n >\!\!\triangleleft \widetilde{SO_q}(n)$ which we understand now as the dilaton-extended orthogonal frame bundle of $\mathbb{R}_q^n$. We need the more general theory of frame resolutions, however, to accommodate the other versions of $\mathbb{R}_q^n$ associated to other quantum groups. One similarly has Minkowski versions $\mathbb{R}_q^{1,3} >\!\!\triangleleft \widetilde{SO_q}(1,3)$ in [20]. This also has a spinorial version where $\mathbb{R}_q^{1,3} = M_q(2)$ the space of 2×2 braided hermitian matrices [23–25]; their spinorial bosonisation $P = M_q(2) >\!\!\triangleleft \widetilde{SU_q(2) \bowtie SU_q(2)}$ is computed explicitly in [26] and can be viewed as a double cover of the $SO_q(1,3)$ frame resolution. Finally, these Euclidean and Minkowski space braided groups have known quantum metrics. A quantum metric in this context of linear braided groups is defined [3, Definition 10.2.14] as an isomorphism of the mutually dual linear braided groups $V_L^*(R', R) \cong V_L(R', R)$ induced by a linear isomorphism η of the mutually dual generating vector spaces. Here (in our present conventions) the evaluation map $\mathrm{ev} : V_L(R', R) \otimes V_L^*(R', R) \to k$ is provided by the braided R-differentiation operators [27]. The coevaluation for this is the appropriate braided-exponential $\exp_R(p|x)$ as a power series in $V_L^*(R', R) \otimes V_L(R', R)$. We refer to [3, Chapter 10] for an introduction to this 'braided analysis'. Projecting to $V = \ker \epsilon$ (polynomials in the generators with no constant terms), we have an induced bilinear form

$$\tilde{\eta} = \exp_R(\eta(p)|x) - 1$$

as a formal power series in $V \otimes V$. This $\tilde{\eta}$ in Proposition 4.3 then induces a formal quantum metric g in the sense of quantum group Riemannian geometry with the universal differential calculus. Note that these linear braided groups also have more natural nonuniversal differential calculi of the correct classical dimension, in which case one expects the universal g to collapse down to a q-deformation of the usual flat metric corresponding to η.

4.3. Other bosonisations and biproducts

We mention here some different settings for braided groups, to which the same formulae as in Proposition 4.4 apply. Thus, one may have B covariant as a left module under a quasitriangular Hopf algebra $H, \mathcal{R}$ where $\mathcal{R} = \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)} \in H \otimes H$ obeys the axioms in [28]. The bosonisation has a similar form $B{>\!\!\!\triangleleft} H$ except that this time the algebra is the cross product by the given action of B and the coalgebra is the cross coproduct by the coaction [29] $\Delta_L(b) = \mathcal{R}^{(2)} \otimes \mathcal{R}^{(1)}{\triangleright} b$. We have the same result as in Proposition 4.4 but with this form of coaction. Thus,

$$\begin{aligned}\nabla(\mathrm{d}b) = 1 \otimes 1 \otimes b - 1 \otimes b \otimes 1 + b \otimes 1 \otimes 1 - b_{(1)} \otimes Sb_{(2)} \otimes b_{(3)} \\ + b_{\underline{(1)}} A(\mathcal{R}^{(2)}\mathcal{R}'^{(2)})\mathcal{R}^{(1)}{\triangleright} \underline{S} b_{\underline{(2)}} \otimes \mathcal{R}'^{(1)}{\triangleright} b_{\underline{(3)}},\end{aligned}$$

where $\mathcal{R}'$ is a second copy of $\mathcal{R}$.

More generally, we can think of braided groups $B \in {}^H_H\mathcal{M}$, the category of crossed modules or quantum double modules associated to any Hopf algebra H with invertible antipode. (These can also be called Drinfeld–Radford–Yetter or DRY-modules cf. [28,30,31].) Here H both acts and coacts on B in a compatible way (to form effectively an action of Drinfeld's quantum double $D(H)$), and the semidirect product and coproduct or 'biproduct' $B{>\!\!\!\triangleleft} H$ is a Hopf algebra. Conversely, every Hopf algebra projection $\pi : P \to H$ split by a Hopf algebra map is of this form for some braided group B. See cf. [30,32] (the latter paper provided the braided group formulation of this theorem of Radford).

We have just the same formulae as in Proposition 4.4 for $B{>\!\!\!\triangleleft} H$ regarded as a frame resolution of such B. On the other hand, we see that general biproducts $B{>\!\!\!\triangleleft} H$ are equivalent to special cases of the quantum homogeneous principal bundles of Section 4.2, namely those which are trivial and where the trivialisation Φ is a Hopf algebra map.

Proposition 4.5. *Let B be a braided group in ${}^H_H\mathcal{M}$. Then its frame resolution by $P = B{>\!\!\!\triangleleft} H$ and $e(v) = \underline{S} v_{\underline{(1)}} \otimes v_{\underline{(2)}}$ can be viewed as a special case of Proposition* 4.3 *with projection $\pi(b \otimes h) = h\underline{\epsilon}(b)$.*

Proof. We apply Proposition 4.3 to this case. Here $V = \underline{\epsilon}$ since $M = B$ and $\epsilon|_M = \underline{\epsilon}$. Since the coproduct in $B{>\!\!\!\triangleleft} H$ has the semidirect coproduct form $\Delta b = b_{\underline{(1)}} b_{\underline{(2)}}{}^{(\bar{1})} \otimes b_{\underline{(2)}}{}^{(\bar{2})}$, the left coaction $\Delta_L(b) = \pi(b_{(1)}) \otimes b_{(2)} = \pi(b_{\underline{(1)}} b_{\underline{(2)}}{}^{(\tilde{1})}) \otimes b_{\underline{(2)}}{}^{(\tilde{2})} = b^{(\tilde{1})} \otimes b^{(\tilde{2})}$ in the proof of Proposition 4.3 coincides with the given left coaction $\overline{\Delta}_L$ used in the construction of $B \rtimes\!\!\!\!\lhd H$. Hence, in Proposition 3.7,

$$\begin{aligned} e(b) &= \pi(b_{(1)})\theta(b_{(2)}) = b^{(\tilde{1})}\theta(b^{(\tilde{2})}) = b^{(\tilde{1})} S b^{(\tilde{2})}{}_{(1)} \otimes b^{(\tilde{2})}{}_{(2)} \\ &= b^{(\tilde{1})} S(b^{(\tilde{2})}{}_{\underline{(1)}} b^{(\tilde{2})}{}_{\underline{(2)}}{}^{(\tilde{1})}) \otimes b^{(\tilde{2})}{}_{\underline{(2)}}{}^{(\tilde{2})} \\ &= b_{\underline{(1)}}{}^{(\tilde{1})} b_{\underline{(2)}}{}^{(\tilde{1})} S(b_{\underline{(1)}}{}^{(\tilde{2})} b_{\underline{(2)}}{}^{(\tilde{2})(\tilde{1})}) \otimes b_{\underline{(2)}}{}^{(\tilde{2})(\tilde{2})} \\ &= (b_{\underline{(1)}}{}^{(\tilde{1})} b_{\underline{(2)}}{}^{(\tilde{1})} S(b_{\underline{(1)}}{}^{(\tilde{2})(\tilde{1})} b_{\underline{(2)}}{}^{(\tilde{2})(\tilde{1})})) \underline{S} b_{\underline{(1)}}{}^{(\tilde{2})(\tilde{2})} \otimes b_{\underline{(2)}}{}^{(\tilde{2})(\tilde{2})} \\ &= (b_{\underline{(1)}}{}^{(\tilde{1})}{}_{(1)} b_{\underline{(2)}}{}^{(\tilde{1})}{}_{(1)} S(b_{\underline{(1)}}{}^{(\tilde{1})}{}_{(2)} b_{\underline{(2)}}{}^{(\tilde{1})}{}_{(2)})) \underline{S} b_{\underline{(1)}}{}^{(\tilde{2})} \otimes b_{\underline{(2)}}{}^{(\tilde{2})} \\ &= \underline{S} b_{\underline{(1)}} \otimes b_{\underline{(2)}}, \end{aligned}$$

where we use the coproduct of $B \rtimes\!\!\!\!\lhd H$, the form of π, the form of θ from Proposition 4.3, the coproduct again, then that the braided coproduct is covariant under the coaction. Finally, we use the antipode $S(bh) = (S(b^{(\tilde{1})}h))\underline{S}b^{(\tilde{2})}$ of $B \rtimes\!\!\!\!\lhd H$ and the comodule axioms to collapse to an antipode cancellation in H. We obtain the quantum V-bein in Proposition 4.4 as required. □

Actually, among trivial quantum homogeneous principal bundles in Section 4.2, there are two natural cases, namely where π is split by a linear map $\Phi : H \to P$ (intertwining the right coaction of H and Δ_R) such that Φ is either an algebra or a coalgebra map. In these cases Φ is convolution-invertible with $\Phi^{-1} = \Phi \circ S$ or $\Phi^{-1} = S \circ \Phi$, respectively. The biproduct case where Φ is a Hopf algebra map is merely the intersection of these two cases. More generally, one has a theory of cocycle biproducts where H weakly coacts on B (up to a cocycle), and a theory of dual-cocycle biproducts where H only weakly coacts on B (up to a dual cocycle).

4.4. Strict quantum groups as base and the quantum double

As a very particular case of the frame resolution of braided groups in the preceding sections, we take $M = BG_q$ the braided group versions of the usual quantum groups G_q. They have been introduced by the author in [23] and are quotients of *braided matrices* $B_L(R)$ with a matrix of generators $\mathbf{u} = \{u^i{}_j\}$, coproduct and quadratic relations

$$\underline{\Delta} u^i{}_j = u^i{}_k \otimes u^k{}_j, \qquad R\mathbf{u}_1 R_{21} \mathbf{u}_2 = \mathbf{u}_2 R \mathbf{u}_1 R_{21}. \tag{26}$$

Here we use a version left-covariant under the corresponding G_q. On the other hand, one knows from [22] that such relations are also obeyed by certain matrix generators of $U_q(\mathfrak{g})$ so one can view BG_q as certain versions of the algebras $U_q(\mathfrak{g})$ (the deeper reason for this is the braided group self-duality isomorphism $BG_q \cong BU_q(\mathfrak{g})$, see [32]). Therefore, if one wants to view $U_q(\mathfrak{g})$ 'up side down' as some kind of coordinate ring, this is one way to do it and $BG_q \rtimes\!\!\!\!\lhd G_q$ is a frame resolution for it. Actually, $BG_q \rtimes\!\!\!\!\lhd G_q \cong G_q \bowtie G_q$ (see [3]) which is essentially isomorphic to some version of the *dual* of Drinfeld's double $D(U_q(\mathfrak{g}))$.

However, the version with BG_q explicitly expresses this dual of the quantum double in the form of Proposition 4.4, with $M = BG_q$ as the base of a trivial quantum principal bundle. We note also that these same bosonisations $BG_q{>\!\!\!\triangleleft} G_q$ have been considered before in [24], as q-deformed Mackey quantisations of a particle moving on BG_q with generalised momentum quantum group $U_q(\mathfrak{g}^*)$.

The general construction behind BG_q is transmutation [33], which associates to any dual quasitriangular Hopf algebra $H, \mathcal{R}$ a braided group $\underline{H}$ covariant under H by the (in our case, left) adjoint action.

Proposition 4.6. *Let $M = \underline{H}$, the braided group version of dual-quasitriangular Hopf algebra H. Then $\underline{H}{>\!\!\!\triangleleft} H$ is a frame resolution of $(\underline{H}, \Omega^1\underline{H})$ by Proposition 4.4, with*

$$V = \ker\epsilon \subset H, \qquad \Delta_L(v) = v_{(1)}Sv_{(3)} \otimes v_{(2)},$$
$$e(v) = \mathcal{R}((Sv_{(2)})^{(\tilde{1})} \otimes v_{(1)})(Sv_{(2)})^{(\tilde{2})} \otimes v_{(3)}$$

Moreover, if H is factorisable with induced linear isomorphism $\mathcal{Q} = \mathcal{R}_{21}\mathcal{R} : H \to H^*$, then we have a quantum cobein

$$f = e_L \circ \mathcal{Q}^{-1} \circ S, \qquad e_L(v) = v_{(1)} \otimes \mathcal{R}((Sv_{(3)})^{(\tilde{1})} \otimes v_{(2)})(Sv_{(3)})^{(\tilde{2})}$$

and hence a quantum metric g.

Proof. (cf. [33]). However, now in a left-handed form, the structure of $\underline{H}$ is

$$\Delta_L h = h_{(1)}Sh_{(3)} \otimes h_{(2)}, \qquad h\underline{\cdot}g = \mathcal{R}(g^{(\tilde{1})} \otimes Sh_{(2)})h_{(1)}g^{(\tilde{2})},$$
$$\underline{S}h = \mathcal{R}((Sh_{(2)})^{(\tilde{1})} \otimes h_{(1)})(Sh_{(2)})^{(\tilde{2})},$$

in terms of the original Hopf algebra structure of H. The braided coproduct and counit coincide with the coproduct and counit of H. Applying this in Proposition 4.4 gives the formula for e. Similarly for e_L. Also, it is well known, cf. [34] that the 'inverse quantum Killing form' $\mathcal{Q} = \mathcal{R}_{21}\mathcal{R}$ is Ad-invariant, and in braided group theory it becomes a braided group homomorphism $\underline{H} \to \underline{H}^*$ by $\mathcal{Q}(w) = (\mathrm{id} \otimes w)\mathcal{Q}$. If we assume that $\mathcal{Q}$ is a linear isomorphism (the so-called factorisable case [34]), then we have the desired left adjoint coaction-invariant 'Killing form' $\eta = \mathcal{Q}^{-1}(S^{-1}f^a) \otimes e_a - 1 \otimes 1$, where $\{e_a\}$ is a basis of H with dual basis $\{f^a\}$. The only subtlety is the antipode needed for the correct coevaluation element $\mathrm{coev} = S^{-1}f^a \otimes e_a \in \underline{H}^* \otimes \underline{H}$, cf. [3, Proposition 9.4.11]. □

The latter 'factorisability' assumption applies to finite-dimensional quantum groups, but it also holds [34] in a formal power-series setting (after allowing suitable square-roots and logarithms of generators) for the standard quantum groups such as G_q and hence BG_q (since this coincides as a linear space with G_q).

Finally, we outline a different frame resolution of BG_q, this time as a left module under a quasitriangular Hopf algebra $H = U_q(\mathfrak{g})$ (as in Section 4.3). Here we view, by definition, that $U_q(\mathfrak{g}) \equiv G_q^*$, i.e. we regard it 'up side down' as the q-deformed coordinate ring of the Drinfeld-dual group with Lie algebra $\mathfrak{g}^*$. Then $BG_q{>\!\!\!\triangleleft} G_q^*\cong D(U_q(\mathfrak{g}))$ (not its dual as before). The general setting here is best covered by using $\underline{H}$ the braided version of a quasitriangular Hopf algebra $H, \mathcal{R}$ in [35] (not dual-quasitriangular as before). One also has $\underline{H}{>\!\!\!\triangleleft} H\cong H \bowtie H$, see [3]. and in the factorisable case one has $\underline{H}{>\!\!\!\triangleleft} H\cong D(H)$, the Drinfeld quantum double [28]. Moreover, $\underline{H} = H$ as an algebra.

Proposition 4.7. *Let $M = H$ be a quasitriangular Hopf algebra. Then $(H, \Omega^1 H)$ has a quantum frame resolution by the 'quantum double' in the form $\underline{H}{>\!\!\!\triangleleft} H$ and the quantum V-bein*

$$V = \ker\epsilon \subset H, \qquad \Delta_L(v) = \mathcal{R}^{(2)} \otimes \mathrm{Ad}_{\mathcal{R}^{(1)}}(v),$$
$$e(v) = X^{(3)}u^{-1}(Sv_{(2)})SX^{(2)} \otimes X^{(1)}v_{(1)},$$

where $X = \mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} \in H^{\otimes 3}$, $u = (S\mathcal{R}^{(2)})\mathcal{R}^{(1)}$ and Ad *is the left quantum adjoint action. The element $\eta = (S\otimes \mathrm{id})(\mathcal{R}_{21}\mathcal{R}) - 1\otimes 1 \in V\otimes V$ is* Ad*-invariant and in the factorisable case induces a quantum metric via the left quantum V-bein*

$$e_L(v) = v_{(1)}X^{(3)} \otimes u^{-1}(SX^{(2)})(Sv_{(2)})X^{(1)}.$$

Proof. We use the braided coproduct and braided antipode of $\underline{H}$ as [3,35]:

$$\underline{\Delta} b = b_{(1)}S\mathcal{R}^{(2)} \otimes \mathrm{Ad}_{\mathcal{R}^{(1)}}(b_{(2)}), \qquad \underline{S}b = u^{-1}(S\mathcal{R}^{(2)})(Sb)\mathcal{R}^{(1)}.$$

Writing out $\mathrm{Ad}_h(g) = h_{(1)}gSh_{(2)}$ explicitly and using Drinfeld's quasitriangularity axioms, one may compute $e(v) = \underline{S}\underline{v_{(1)}} \otimes \underline{v_{(2)}}$ and $e_L(v) = \underline{v_{(1)}} \otimes \underline{S}\underline{v_{(2)}}$ as shown in terms of the structure of H (among many other ways to write these objects). This is a straightforward Hopf-algebra calculation. Meanwhile, Ad-invariance of $(S\otimes\mathrm{id})\mathcal{R}_{21}\mathcal{R}$ is well known and given explicitly in [3, Chapter 2]. One may go on and write the quantum metric $g = e_L(\eta^{(1)})e(\eta^{(2)})$ explicitly as a product of several copies of $\mathcal{R}$. □

Thus, the braided version $BU_q(\mathfrak{g})$, which has the same algebra as $U_q(\mathfrak{g})$, has frame resolution $BU_q(\mathfrak{g}){>\!\!\!\triangleleft} U_q(\mathfrak{g})$. This can be applied to the reduced quantum group enveloping algebras at roots of unity (which are finite-dimensional), or applied in the formal power-series setting of [28]. In another other version of Proposition 4.7 we may take BG_q in place of $BU_q(\mathfrak{g})$ since these are essentially isomorphic in the factorisable case. Then BG_q has a frame resolution by $BG_q{>\!\!\!\triangleleft} U_q(\mathfrak{g})$. Likewise, we have a version of Proposition 4.6 where we replace BG_q by a suitable (right Ad-action covariant) version of $BU_q(\mathfrak{g})$.

There are many other braided groups beyond those discussed above. For example, it is obvious from Lusztig's book [36] that in his approach to the structure of $U_q(\mathfrak{g})$ one effectively views $U_q(n_+)$ as a braided group with phase-factor braid statistics as in [21]. The above results provide a step towards a 'quantum group Riemannian geometry' of such objects as well, albeit far removed from our original physical motivation of q-deforming usual geometry.

Acknowledgements

These results were obtained on a visit to Padstow, Cornwall, in May 1997.

Appendix A. Braided group 'diagrammatic' Riemannian geometry

Here we give the formalism developed above in a different 'diagrammatic' setting of braided group gauge theory [9,10], i.e. where the gauge group has braid statistics (this should not be confused with Section 4 where we gave some examples where the base M was a braided group). In particular, the author in [10] developed principal bundles, connections, associated bundles, etc. at the level of braid and tangle diagrams, which theory we 'update' now to include the elements of Riemannian geometry above. As well as being more general and having potentially different examples than the quantum group case (cf. the anyonic (or $\mathbb{Z}_n$-graded) gauge theory in [10]), the diagrammatic theory provides a different style of proofs which, for trivial braid statistics, reduces to the quantum group gauge theory. For super ($\mathbb{Z}_2$-graded) gauge theory we just take bose-fermi statistics, so the formulae for super-quantum group Riemannian geometry can be read off from these diagrams. In general, we work in a braided category [37] where, for any two objects, there is a braiding $\Psi = \times$ implementing their exchange. We denote $\Psi^{-1} = \times$. Extending this notation, algebra products are denoted $\curlyvee$ and coproducts or coactions denoted $\curlywedge$. Maps are provided by 'wiring' outputs into inputs, with maps flowing generally downwards. The unit object $\underline{1}$ for the tensor product is denoted by omission. This is the 'diagrammatic braided group theory' introduced in [35]. See [2,3].

Thus, we consider an algebra M in a braided category, the differential calculus $\Omega^1 M$ defined diagrammatically and a braided group principal bundle P, B. Here B is a braided group or Hopf algebra with braid statistics [35]. If V is a right B-comodule in the braided category, we have an associated bundle $\mathcal{E} = (P \otimes V)^B$ as before, where fixed points are defined categorically as equalisers and where $P \otimes V$ has the braided tensor product coaction. This theory is in [10]. We assume that our braided category has direct sums and appropriate flatness properties, as explained in [10], and adopt the corresponding abuses of notation. To this we add:

Lemma A.1. *Right strongly tensorial $\theta : V \to P\Omega^n M$ are in 1–1 correspondence with left M-module morphisms $\mathcal{E} \to \Omega^n M$.*

Proof. This is shown in Fig. 1 for $n = 1$ (the general case looks just the same). We (a) apply the braided coaction $P \to P \otimes B$ as shown and find that it acts trivially, hence the morphism factors through $\Omega^1 M$. Conversely, (b) constructs $\theta : V \to P\Omega^n M$ given $s : \mathcal{E} \to \Omega^n M$. The proofs follow exactly the steps for the $n = 0$ case in [10], so we omit the detailed diagrammatic verification. The morphism $\eta : \underline{1} \to B$ is playing the role of the unit 'element' of the braided group (and should not be confused with the local metric η in the main sections of the paper). □

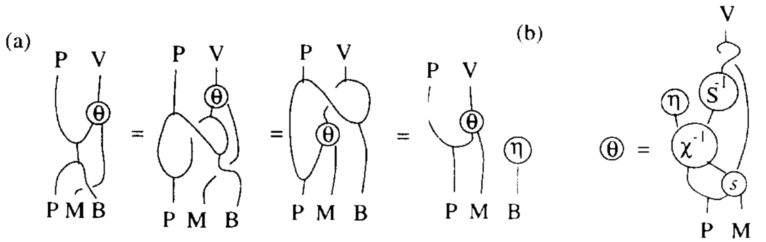

Fig. 1. Proof of Lemma A.1.

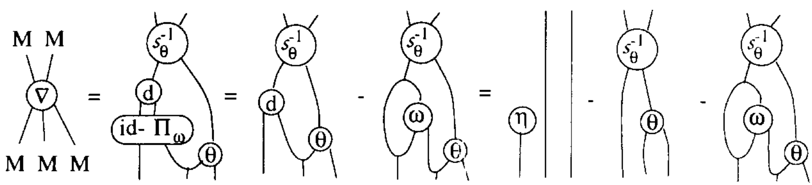

Fig. 2. Diagrammatic form of ∇.

We then define a braided frame resolution as a pair P, θ such that P is a braided principal bundle and the morphism $s_\theta = (\cdot \otimes \mathrm{id})(\mathrm{id} \otimes \theta)$ is an isomorphism. We then induce similar isomorphisms for $\Omega^1 M \otimes_M \Omega^1 M$ by $\mathrm{id} \otimes s_\theta$.

Proposition A.2. *Given a braided frame resolution of M and a connection ω, we define $\nabla : \Omega^1 M \to \Omega^1 M \otimes_M \Omega^1 M$ as the covariant derivative $D = (\mathrm{id} - \Pi_\omega)\mathrm{d}$ viewed under the above isomorphisms. It is computed in Fig. 2 and is a derivation with respect to multiplication in the first factor.*

Proof. This is shown in Fig. 2, using the expression for $\mathrm{id} - \Pi_\omega$ in [10]. The derivation formula is then immediate from the final result for ∇. The unmarked ⋂ denotes the coaction $\Delta_R : P \to P \otimes B$. □

Thus, we are able to proceed along the same lines as in the quantum group case, with similar results. We let

$$\Delta_L = (S^{-1} \otimes \mathrm{id})\Psi^{-1} \circ \Delta_R,$$

which [2] makes P into a braided left $\bar{B}$-module algebra in the category with reversed braiding. Here $\bar{B}$ denotes B with the opposite product $\cdot \circ \Psi^{-1}$ and is [2] a braided group in the category with reversed braiding. We define $\bar{D} = (\mathrm{id} - \bar{\Pi}_\omega)\,\mathrm{d}$ on P, where $\bar{\Pi}_\omega$ is the left-handed version (the mirror reflection) of the diagram for Π_ω in [10].

Proposition A.3. *The following are equivalent: (a) A connection ω is left strong, (b) The equality in Fig.* 3(b) *holds, (c) The equality in Fig.* 3(c) *holds. In this case, $\bar{D}$ preserves right strongly tensorial forms on P.*

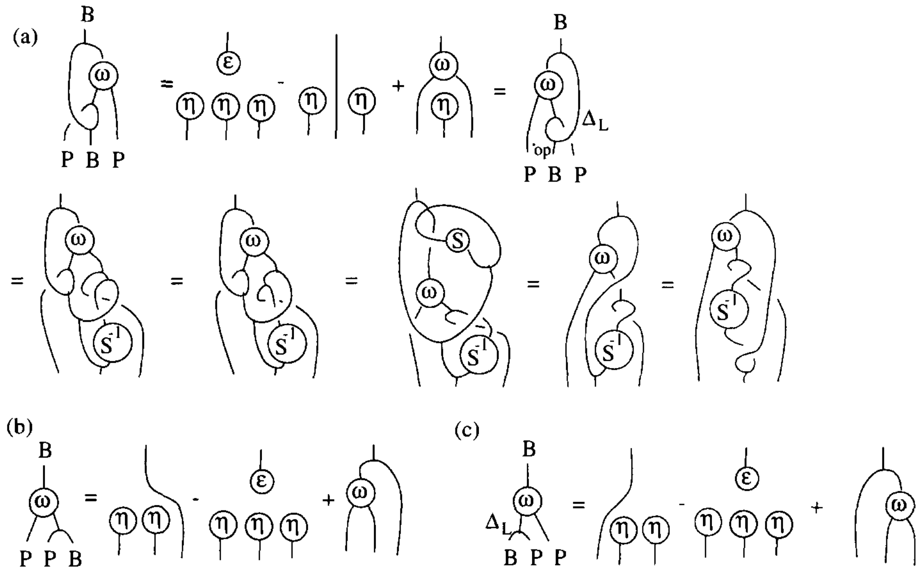

Fig. 3. Equivalent versions of the strongness condition for ω.

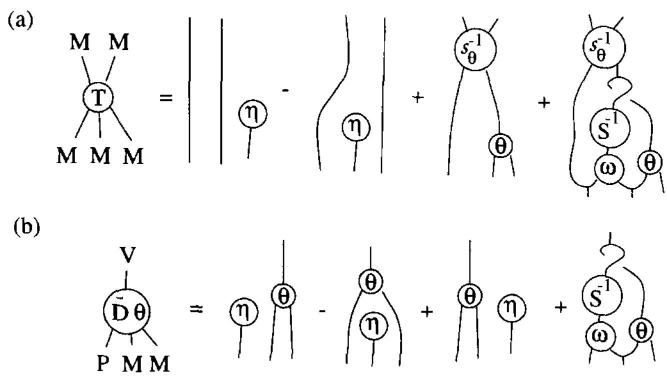

Fig. 4. Braided torsion tensor.

Proof. The condition for ω to be left strong (to preserve left strongly tensorial forms) is given in [10]. Applying $P \otimes_M (\,)$ to this diagram and making a product in P, one finds that it factors through $\chi : P \otimes_M P \to P \otimes B$. Cancelling this gives the first equality in Fig. 3(a) as the condition for left strongness. The left-hand side of this is, however, equal to its mirror image (shown on the right in Fig. 3(a)). The proof of this is the lower line in Fig. 3(a). We first insert a trivial 'antipode loop'. The next equality is the comodule property of Δ_R. We then use Ad-invariance of ω, cancel a resulting antipode-loop and finally rearrange to recognize the right-hand side of the upper line in Fig. 3(a). Parts (b) and (c) immediately follow as equivalent to these two versions of the left-strongness conditions: We precompose with the coproduct of B, apply the antipode to the free leg thus created and join up as a product

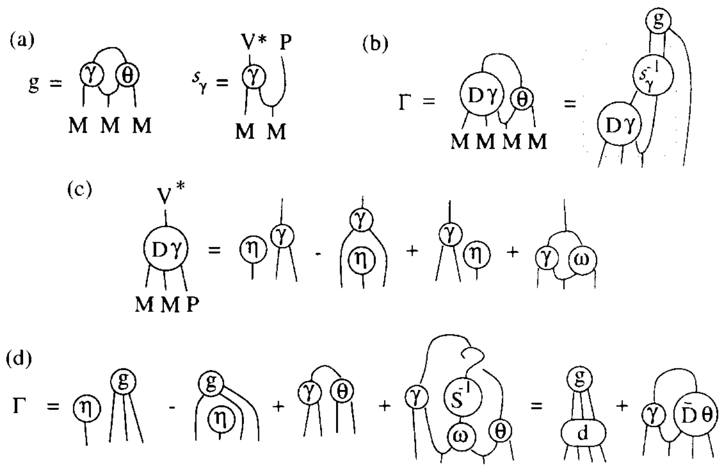

Fig. 5. Braided metric and its cotorsion.

in B from the appropriate side in such a way as to create an antipode loop cancellation in each term. As a corollary, if ω is such that D preserves left strongly tensorial forms, then $\bar{D}$ preserves right strongly tensorial forms since the condition for the latter is the mirror image of the condition for the former. We use the mirror image of the proof of [10, Proposition 4.2] (this part of the proof does not actually require the Galois condition). □

Proposition A.4. *We define the braided torsion tensor:* $\Omega^1 M \to \Omega^1 M \otimes_M \Omega^1 M$ *as* $\mathrm{d}-\nabla$. *This is shown in Fig.* 4(a). *It corresponds under the above isomorphisms to* $\bar{D}\theta$ *as a right strongly tensorial* 2*-form, shown in Fig.* 4(b).

Proof. The proof follows the quantum group case. The exterior derivative has the identical form (but written diagrammatically) and we subtract ∇. We also write the coaction on P in $(P \otimes V)^B$ in ∇ as a coaction on V (see [10, Proposition 4.4]). For the second part, we graft on a product with $P \otimes_M$ from the left and cancel a copy of χ through which the diagram factors. The resulting diagram is same as grafting on a product with P from the left to $\bar{D}\theta$ shown in Fig. 4(b). This is what is required according to Lemma A.1. Finally, $\bar{D}\theta = (\mathrm{id} - \bar{\Pi}_\omega)\,\mathrm{d}\theta$ when this is computed in a similar way to Proposition A.3. □

Next, we can formulate a metric as, by definition, corresponding to γ a left strongly tensorial form making (P, B, V^*, γ) a (left-handed) frame resolution. The associated metric $g : \underline{1} \to \Omega^1 M \otimes_M \Omega^1 M = \Omega^2 M$ is given in Fig. 5(a), along with the associated left-handed version of the isomorphism $s_\gamma : (V^* \otimes P)^B \cong \Omega^1 M$. This assumes that V is rigid in the sense of a dual with coevaluation $\cap : \underline{1} \to V^* \otimes V$. More generally, one should work instead with (P, B, V, θ_L) a left-handed frame resolution and an invariant morphism $G : \underline{1} \to V \otimes V$ (say). Then $g = \cdot \circ (\theta_L \otimes \theta) \circ G$. The diagrams in this case are very similar, so we will not repeat them explicitly.

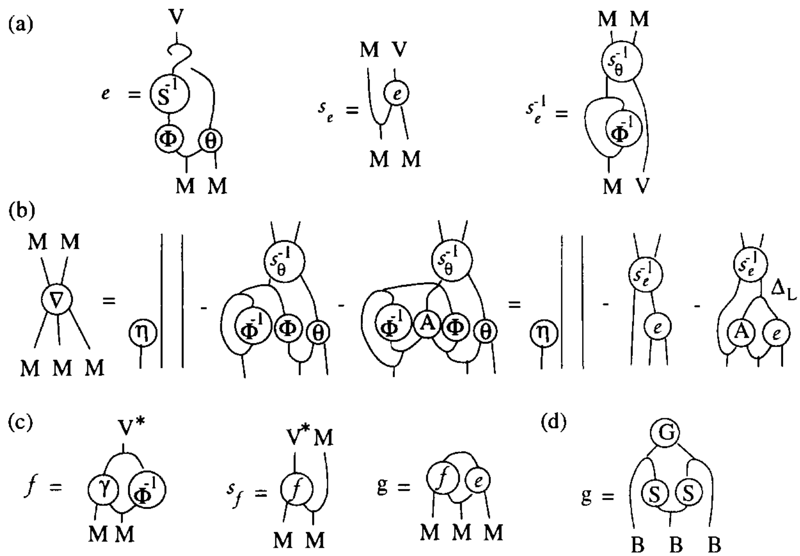

Fig. 6. Local picture in terms of braided V-bein and cobein, and gauge field A.

Proposition A.5. *The cotorsion form Γ defined as corresponding to $D\gamma$ under* $\mathrm{id}\otimes s_\theta$ *coincides with* $(T_\gamma\otimes\mathrm{id})\circ g$, *where T_γ is the torsion of γ computed in the dual frame resolution evaluated on g. Moreover,* $\Gamma = \mathrm{d}g + (\mathrm{id}\otimes T)\circ g$.

Proof. The first part is shown in Fig. 5(b) and follows at once from the definition of the metric g in terms of γ. Here T_γ is shown in the dotted box as obtained from $D\gamma$ via s_γ^{-1}. Fig. 5(c) gives the explicit form of $D\gamma$ as a special case of [10]. Part (d) then gives the explicit form of Γ given by combining $D\gamma$ with θ as in part (b). Similarly combining $\bar{D}\theta$ from Fig. 5 with γ gives an expression with similar last two terms; comparing them we see that the difference is precisely $\mathrm{d}g$. □

Finally, if the frame resolution bundle is trivial, with trivialisation $\Phi : B \to P$, we define $e : V \to \Omega^1 M$ corresponding to θ via Fig. 6(a). Also shown is the induced isomorphism $s_e : M\otimes V\cong\Omega^1 M$ by $s_e = s_\theta\circ\theta_{\mathcal{E}}$, where $\theta_{\mathcal{E}} : M\otimes V\cong\mathcal{E}$ is given in [10]. This makes it clear that s_θ invertible is equivalent to s_e invertible. Similarly, the existence of a dual frame resolution γ is equivalent to $f : V^* \to \Omega^1 M$ such that s_f is invertible (see Fig. 6(c)). Similarly, we already know [10] that strong connections ω correspond to gauge fields $A : B \to \Omega^1 M$ via $\omega = \Phi^{-1} * A * \Phi + \Phi^{-1} * \mathrm{d}\Phi$. Putting this into ∇ from Fig. 2 and moving the coaction on P over to Δ_L in V (as in the preceding proof) gives the resulting covariant derivative ∇ in terms of e, A as shown in Fig. 6(b). In fact, the resulting 'local' formulae take the same form as in the quantum group case at the end of Section 3 if we use the convolution product notation. One has similarly that $D\gamma$ and $\bar{D}\theta$ correspond to $D_A f$ and $\bar{D}_A e$ as in Proposition 3.7, written diagrammatically as morphisms. The formulae for a left-handed frame resolution e_L are similar.

We see that braided theory goes through along the lines of the quantum group case. One has braided versions of all the examples in Section 4 as well: braided groups, braided homogeneous spaces and braided cross products such as $\mathbb{C}_q^2{>\!\!\!\triangleleft} BGL_q(2)$ in the parallel way. For example, the braided analogue of Proposition 4.1 is $M = B$ a braided group, $V = \ker \underline{\epsilon}$ (defined now as an equaliser), and $\theta = (\underline{S} \otimes \mathrm{id}) \circ \underline{\Delta}$. There is also a left-handed $\theta_L = (\mathrm{id} \otimes \underline{S}) \circ \underline{\Delta}$ and hence if there is an invariant morphism $G : \underline{1} \to V \otimes V$ (say) we have a braided metric as shown in Fig. 6(d). The general braided theory is, however, potentially better behaved as regards the Ad bundle and other properties than the quantum group theory through the imposition of natural 'braided commutativity', see [10]. Moreover, the braided setting allows one to read off the $\mathbb{Z}_2$ and $\mathbb{Z}_n$-graded versions of the theory by inserting the relevant braid statistics phase factor at each braid crossing.

References

[1] S. Majid, Hopf algebras for physics at the Planck scale, J. Classical Quantum Gravity 5 (1988) 1587–1606.
[2] S. Majid, Beyond supersymmetry and quantum symmetry (an introduction to braided groups and braided matrices), in: M.-L. Ge, H.J. de Vega (Eds.), Quantum Groups, Integrable Statistical Models and Knot Theory, World Scientific, Singapore 1993, pp. 231–282.
[3] S. Majid, Foundations of Quantum Group Theory, Cambridge University Press, Cambridge, 1995.
[4] T. Brzeziński, S. Majid, Quantum group gauge theory on quantum spaces, Comm. Math. Phys. 157 (1993) 591–638; Erratum 167 (1995) 235.
[5] P. Hajac, Strong connections on quantum principal bundles, Comm. Math. Phys. 182 (1996) 579–617.
[6] S. Majid, Some remarks on quantum and braided group gauge theory, in: Banach Center Publications, vol. 40, 1997, pp. 335–349.
[7] T. Brzeziński, Translation map in quantum principal bundles, J. Geom. Phys. 20 (1996) 349–370.
[8] T. Brzeziński, S. Majid, Quantum differentials and the q-monopole revisited, Acta Appl. Math. 54 (1998) 185–232.
[9] T. Brzezinski, S. Majid, Coalgebra bundles, Comm. Math. Phys. 191 (1998) 467–492.
[10] S. Majid, Diagrammatics of braided group gauge theory, preprint, 1995.
[11] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry, Wiley/Interscience, New York, 1969.
[12] I. Kolar, P. Michor, J. Slovak, Natural Operations in Differential Geometry, Springer, Berlin, 1991.
[13] A. Connes, Noncommutative Geometry, Academic Press, New York, 1995.
[14] P. Hajac, S. Majid, Projective module description of the q-monopole, Commun. Math. Phys. (1999), to appear.
[15] Heckenberger, K. Schmüdgen, Levi-civita connections on the quantum groups $SL_q(N)$, $O_q(N)$ and $Sp_q(N)$, preprint.
[16] M. Durdevic, General frame structures on quantum principal bundles, preprint.
[17] H.-J. Schneider, Principal homogeneous spaces for arbitrary Hopf algebras, Isr. J. Math. 72 (1990) 167–195.
[18] S.L. Woronowicz, Differential calculus on compact matrix pseudogroups (quantum groups), Comm. Math. Phys. 122 (1989) 125–170.
[19] P. Podles, Quantum spheres, Lett. Math. Phys. 14 (1987) 193–202.
[20] S. Majid, Braided momentum in the q-Poincaré group, J. Math. Phys. 34 (1993) 2045–2058.
[21] S. Majid, $\mathbb{C}$-statistical quantum groups and Weyl algebras, J. Math. Phys. 33 (1992) 3431–3444.
[22] L.D. Faddeev, N.Yu. Reshetikhin, L.A. Takhtajan, Quantization of Lie groups and Lie algebras, Leningrad Math. J. 1 (1990) 193–225.
[23] S. Majid, Examples of braided groups and braided matrices, J. Math. Phys. 32 (1991) 3246–3253.
[24] S. Majid, The quantum double as quantum mechanics, J. Geom. Phys. 13 (1994) 169–202.
[25] U. Meyer, q-Lorentz group and braided coaddition on q-Minkowski space, Comm. Math. Phys. 168 (1995) 249–264.

[26] S. Majid, U. Meyer, Braided matrix structure of q-Minkowski space and q-Poincaré group, Z. Phys. C 63 (1994) 357–362.
[27] S. Majid, Free braided differential calculus, braided binomial theorem and the braided exponential map, J. Math. Phys. 34 (1993) 4843–4856.
[28] V.G. Drinfeld, Quantum groups, in: A. Gleason (Ed.), Proceedings of the ICM, AMS, RI, 1987, pp. 798–820.
[29] S. Majid, Doubles of quasitriangular Hopf algebras, Comm. Algebra 19 (11) (1991) 3061–3073.
[30] D. Radford, The structure of Hopf algebras with a projection, J. Algebra 92 (1985) 322–347.
[31] D.N. Yetter, Quantum groups and representations of monoidal categories, Math. Proc. Cambridge Philos. Soc. 108 (1990) 261–290.
[32] S. Majid, Braided matrix structure of the Sklyanin algebra and of the quantum Lorentz group, Comm. Math. Phys. 156 (1993) 607–638.
[33] S. Majid, Braided groups, J. Pure Appl. Algebra 86 (1993) 187–221.
[34] N.Yu. Reshetikhin, M.A. Semenov-Tian-Shansky, Quantum R-matrices and factorization problems, J. Geom. Phys. 5 (1988) 533.
[35] S. Majid, Braided groups and algebraic quantum field theories, Lett. Math. Phys. 22 (1991) 167–176.
[36] G. Lusztig. Introduction to Quantum Groups, Birkhäuser, Basel, 1993.
[37] A. Joyal, R. Street, Braided monoidal categories, Mathematics Reports 86008, Macquarie University, 1986.